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Tableau Calculus
Tableau Calculus
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary
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Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  ⇝ idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic
Automation

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\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]
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negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:
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Negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

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\neg(p \lor q) \lor (\neg p \lor \neg q)
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Negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

\[\neg(p \lor q) \lor (\neg p \lor \neg q)\]

\[(\neg p \land \neg q) \lor (\neg p \lor \neg q)\]
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
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Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]
Simple Tableau

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches

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Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

\[\neg p \land \neg q \quad \neg p \quad \neg q\]

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
- compare: truth table

<table>
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<th>(I(q))</th>
<th>(I(\neg p))</th>
<th>(I(\neg q))</th>
<th>(I(p \lor q))</th>
<th>(I(\neg p \lor \neg q))</th>
<th>(I((p \lor q) \rightarrow (\neg p \lor \neg q)))</th>
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Simple Tableau with Contradiction

\[ (\neg p \lor q) \land p \land \neg q \]
Simple Tableau with Contradiction

$$(\neg p \lor q) \land p \land \neg q$$

$$(\neg p \lor q)$$

$$p$$

$$\neg q$$
Simple Tableau with Contradiction

\( \neg p \lor q \land p \land \neg q \)

\( \neg p \lor q \)

\( p \)

\( \neg q \)

\( \neg p \quad q \)
Simple Tableau with Contradiction

\[
(\neg p \lor q) \land p \land \neg q
\]

\[
\neg p \lor q
\]

\[
p
\]

\[
\neg q
\]

\[
\neg p \qquad q
\]

if a branch contains an atomic contradiction (clash), we call this branch closed.

A tableau is closed if all its branches are.

A complete tableau without open branches shows the formula's unsatisfiability.
Simple Tableau with Contradiction

\[(\lnot p \lor q) \land p \land \lnot q\]

- \(\lnot p \lor q\)
  - \(p\)
    - \(\lnot q\)
      - \(\lnot p\)
      - \(q\)

\(\bot\)

- if a branch contains an atomic contradiction (clash), we call this branch closed
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

- \(\neg p \lor q\)
  - \(p\)
    - \(\neg q\)
      - \(\neg p\)
      - \(q\)

- \(\bot\)
  - \(\bot\)

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
Simple Tableau with Contradiction

\[ (\neg p \lor q) \land p \land \neg q \]

- \( \neg p \lor q \)
- \( p \)
- \( \neg q \)
- \( \neg p \)
- \( q \)

\[ \bot \]

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
- a complete tableau without open branches shows the formula's unsatisfiability
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- \(\neg p\)
- \(\neg q\)

- given an open branch, we can construct a model
Constructing a Model from the Tableau

\[ \neg p \land \neg q \lor \neg p \lor \neg q \]

\[ \neg p \land \neg q \]
\[ \neg p \]
\[ \neg q \]

- given an open branch, we can construct a model
- let \( I(p) = \text{false} \) and let \( I(q) = \text{false} \)
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- given an open branch, we can construct a model
- let \(I(p)\) = false and let \(I(q)\) = false
- let \(I(p)\) = false (\(I(q)\) is irrelevant since not in the branch, default assignment false)
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- given an open branch, we can construct a model
- let \(I(p)=false\) and let \(I(q)=false\)
- let \(I(p)=false\) (\(I(q)\) is irrelevant since not in the branch, default assignment false)
- let \(I(q)=false\) (\(I(p)\) is irrelevant since not in the branch, default assignment false)
Propositional Tableau

- not always exponentially many combinations have to be checked (as opposed to truth table method)
- branches can be built one after the other $\Rightarrow$ only polynomial space needed
- if we care about satisfiability we can stop after constructing the first complete open branch
Construction with only one Branch in Memory

\((\neg p \lor q) \land p \land q\)
Construction with only one Branch in Memory

\[ (\neg p \lor q) \land p \land q \]
\[ \neg p^{1a} \lor q^{1b} \]
\[ p \]
\[ q \]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\[ (\neg p \lor q) \land p \land q \]
\[ \neg p^{1a} \lor q^{1b} \]
\[ p \]
\[ q \]
\[ \neg p^{1a} \]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\[
\neg p \lor q \land p \land q
\]

\[
\neg p^{1a} \lor q^{1b}
\]

\[
p
\]

\[
q
\]

\[
\neg p^{1a}
\]

\[
\bot^{1a}
\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

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From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts? NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory “by itself”.

- tableau represents an element of the domain (plus its “environment”).
From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts? NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory "by itself".

- tableau represents an element of the domain (plus its "environment")
- tableau branch: finite set of propositions of the form $C(a), r(a, b)$
- for existential quantifiers, new domain elements are introduced
- universal quantifiers propagate formulae (=concept expressions) to neighboring elements
Agenda

- Basic Idea of the Tableau Calculus
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Propositional Logic – Some Logical Equivalences

- We aim at negations being present only in front of atomic concepts

\[
\begin{align*}
\phi \land \psi & \equiv \psi \land \phi \\
\phi \lor \psi & \equiv \psi \lor \phi \\
\phi \land (\psi \land \omega) & \equiv (\phi \land \psi) \land \omega \\
\phi \lor (\psi \lor \omega) & \equiv (\phi \lor \psi) \lor \omega \\
\phi \land \phi & \equiv \phi \\
\phi \lor \phi & \equiv \phi \\
\phi \land (\psi \lor \phi) & \equiv \phi \\
\phi \lor (\psi \land \phi) & \equiv \phi \\
\phi \lor (\psi \lor \phi) & \equiv \phi \\

\phi \rightarrow \psi & \equiv \neg \phi \lor \psi \\
\phi \leftrightarrow \psi & \equiv (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \\
\neg (\phi \land \psi) & \equiv \neg \phi \lor \neg \psi \\
\neg (\phi \lor \psi) & \equiv \neg \phi \land \neg \psi \\
\neg \neg \phi & \equiv \phi \\
\phi \land (\psi \lor \omega) & \equiv (\phi \lor \psi) \land (\phi \lor \omega) \\
\phi \lor (\psi \land \omega) & \equiv (\phi \land \psi) \lor (\phi \land \omega)
\end{align*}
\]
Further Logical Equivalences

\( \neg (C \cap D) \leftrightarrow \neg C \sqcup \neg D \)

\( \neg (D \sqcup D) \leftrightarrow \neg C \cap \neg D \)

\( \neg \neg C \leftrightarrow C \)

\( \neg (\forall r. C) \leftrightarrow \exists r. (\neg C) \)

\( \neg (\exists r. C) \leftrightarrow \forall r. (\neg C) \)

\( \neg (\leq n\; s. C) \leftrightarrow \geq n + 1\; s. C \)

\( \neg (\geq n\; s. C) \leftrightarrow \leq n - 1\; s. C, \quad n \geq 1 \)

\( \neg (\geq 0\; s. C) \leftrightarrow \bot \)

- apply these rules iteratively until none can be applied any more

\( \leftrightarrow \) equivalent concept in negation normal form
**NNF Transformation**

recursive definition of an NNF transformation:

\[ \text{if } C \text{ atomic:} \]
\[ \text{NNF}(C) := C \quad \text{NNF}(\neg C) := \neg C \]

\[ \text{otherwise:} \]
\[ \text{NNF}(\neg\neg C) := \text{NNF}(C) \]
\[ \text{NNF}(C \cap D) := \text{NNF}(C) \cap \text{NNF}(D) \quad \text{NNF}(\neg(C \cap D)) := \text{NNF}(\neg C) \sqcup \text{NNF}(\neg D) \]
\[ \text{NNF}(C \cup D) := \text{NNF}(C) \cup \text{NNF}(D) \quad \text{NNF}(\neg(C \cup D)) := \text{NNF}(\neg C) \sqcap \text{NNF}(\neg D) \]
\[ \text{NNF}(\forall r.C) := \forall r.\text{NNF}(C) \quad \text{NNF}(\neg(\forall r.C)) := \exists r.\text{NNF}(\neg C) \]
\[ \text{NNF}(\exists r.C) := \exists r.\text{NNF}(C) \quad \text{NNF}(\neg(\exists r.C)) := \forall r.\text{NNF}(\neg C) \]
\[ \text{NNF}(\leq n s.C) := \leq n s.\text{NNF}(C) \quad \text{NNF}(\neg(\leq n s.C)) := \geq n + 1 s.\text{NNF}(C) \]
\[ \text{NNF}(\geq n s.C) := \geq n s.\text{NNF}(C) \quad \text{NNF}(\neg(\geq n s.C)) := \leq n - 1 s.\text{NNF}(C) \]
\[ \quad \text{if } n \geq 1 \]
\[ \text{NNF}(\geq 0 s.C) := \top \quad \text{NNF}(\neg(\geq 0 s.C)) := \bot \quad \text{otherwise} \]
NNF Transformation – Example

\[
\begin{align*}
NNF(\neg (\neg C \sqcap (\neg D \sqcup E))) &= NNF(\neg\neg C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
&= NNF(C) \sqcup NNF(\neg(\neg D \sqcup E)) \\
&= C \sqcup NNF(\neg(\neg D \sqcup E)) \\
&= C \sqcup (NNF(\neg\neg D) \sqcap NNF(\neg E)) \\
&= C \sqcup (NNF(D) \sqcap NNF(\neg E)) \\
&= C \sqcup (D \sqcap NNF(\neg E)) \\
&= C \sqcup (D \sqcap \neg E)
\end{align*}
\]
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Tableau for $\mathcal{ALC}$ Concepts

- tableau for a propositional formula $\alpha$: one element, labeled with subformulae of $\alpha$
- tableau for an $\mathcal{ALC}$ concept $C$: graph (more precisely: tree) where the nodes are labeled with subformulae of $C$
- root labeled with $C$
- represents model for $C$ (if complete and clash-free)
- non-root nodes are enforced by existential quantifiers

**Definition**

Let $C$ be an $\mathcal{ALC}$ concept, $\text{SF}(C)$ the set of all subformulae of $C$ and $\text{Rol}(C)$ the set of all roles occurring in $C$. A tableau for $C$ is a tree $G = \langle V, E, L \rangle$, with nodes $V$, edges $E \subseteq V \times V$ and a labeling function $L$ with $L: V \rightarrow 2^{\text{SF}(C)}$ and $L: V \times V \rightarrow 2^{\text{Rol}(C)}$. 

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Properties of the $\mathcal{ALC}$ Tableau Algorithm

- the algorithm is specified as a set of rules
- every rule breaks down a complex concept into its parts
- rules applicable in any order
- the algorithm is non-deterministic (due to disjunction)
- check for atomic contradictions

Tableau algorithm for checking satisfiability of $\mathcal{ALC}$ concepts

**Input:** an $\mathcal{ALC}$ concept in NNF

**Output:**
- `true` if there is a clash-free tableau where no more rules can be applied
- `false` otherwise (tableau closed)
Tableau Rules for $\mathbb{ALC}$ Concepts

\(\Box\)-rule: For an arbitrary \(v \in V\) mit \(C \cap D \in L(v)\) and 
\[\{C, D\} \not\subseteq L(v),\] 
let \(L(v) := L(v) \cup \{C, D\}\).

\(\sqcup\)-rule: For an arbitrary \(v \in V\) with \(C \sqcup D \in L(v)\) and 
\[\{C, D\} \cap L(v) = \emptyset,\] 
choose \(X \in \{C, D\}\) and let 
\(L(v) := L(v) \cup \{X\}\).

\(\exists\)-rule: For an arbitrary \(v \in V\) with \(\exists r.C \in L(v)\) such that 
there is no \(r\)-successor \(v'\) of \(v\) with \(C \in L(v')\), 
let \(V = V \cup \{v'\}, E = E \cup \{(v, v')\}, L(v') := \{C\}\) and 
\(L(v, v') := \{r\}\) for \(v'\) a new node.

\(\forall\)-rule: For arbitrary \(v, v' \in V, v' r\)-neighbor of \(v,\) 
\(\forall r.C \in L(v)\) and \(C \notin L(v')\), let 
\(L(v') := L(v') \cup \{C\}\).

- a node \(v'\) is an \(r\)-neighbor of a node \(v\) if \((v, v') \in E\) and \(r \in L(v, v')\)
Tableau Rules for $\mathcal{ALC}$ Concepts

$\exists$-rule: For an arbitrary $v \in V$ with $\exists r.C \in L(v)$ such that there is no $r$-successor $v'$ of $v$ with $C \in L(v')$, let $V := V \cup \{v'\}$, $E := E \cup \{(v, v')\}$, $L(v') := \{C\}$ and $L(v, v') := \{r\}$ for $v'$ a new node.

$\forall$-rule: For arbitrary $v, v' \in V$, $v'$ $r$-neighbor of $v$,

$\forall r.C \in L(v)$ and $C \notin L(v')$, let $L(v') := L(v') \cup \{C\}$.

- a node $v'$ is an $r$-neighbor of a node $v$ if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
Tableau Rules for $\mathcal{ALC}$ Concepts

$\square$-rule: For an arbitrary $v \in V$ with $C \cap D \in L(v)$ and
$
\{C, D\} \not\subseteq L(v),
$ let $L(v) := L(v) \cup \{C, D\}$.

$\Box$-rule: For an arbitrary $v \in V$ with $C \sqcup D \in L(v)$ and
$
\{C, D\} \cap L(v) = \emptyset,
$ choose $X \in \{C, D\}$ and let
$L(v) := L(v) \cup \{X\}$.

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$V = V \cup \{v'\}$, $E = E \cup \{(v, v')\}$, $L(v') := \{C\}$ and
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$\forall$-rule: For arbitrary $v, v' \in V$, $v'$ $r$-neighbor of $v$,
$
\forall r.C \in L(v) \text{ and } C \notin L(v'),
$ let $L(v') := L(v') \cup \{C\}$.

- a node $v'$ is an $r$-neighbor of a node $v$ if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
- choice of disjunction: “don’t know” non-determinism
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{C\} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r.(\neg A \sqcap \forall r.(\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B \} \]

\[ L(w) = \{ \neg A \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \lor \exists r. B) \land \exists r. \neg A \land \forall r. (\neg A \land \forall r. (\neg B \lor A)) \]

\[
\begin{align*}
L(u) & = \{ C, \exists r. (A \lor \exists r. B), \\
& \quad \exists r. \neg A, \forall r. (\neg A \land \forall r. (\neg B \lor A)) \} \\
L(v) & = \{ A \lor \exists r. B, \neg A, \forall r. (\neg B \lor A) \} \\
L(w) & = \{ \neg A \}
\end{align*}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \}
\]

\[
L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A) \}
\]

\[
L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), A \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{C, \exists r. (A \sqcup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A))\}
\]

\[
L(v) = \{A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B\}
\]

\[
L(w) = \{\neg A, \forall r. (\neg B \sqcup A)\}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} 
\]

\[
L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A) \} 
\]

\[
L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} 
\]

\[
L(x) = \{ B \} 
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[
L(u) = \{ C, \exists r. (A \cup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \\
L(v) = \{ A \cup \exists r. B, \neg A, \forall r. (\neg B \cup A), \times, \exists r. B \} \\
L(w) = \{ \neg A, \forall r. (\neg B \cup A) \} \\
L(x) = \{ B, \neg B \cup A \} 
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \}
\]

\[
L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \}
\]

\[
L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \}
\]

\[
L(x) = \{ B, \neg B \sqcup A, \neg B \}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \color{red} \times, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]

\[ L(x) = \{ B, \neg B \sqcup A, \color{red} \times, \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
\begin{array}{c}
L(u) = \{C, \exists r. (A \sqcup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A))\}
\end{array}
\]

\[
\begin{array}{c}
L(v) = \{A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \neg B, \exists r. B\}
\end{array}
\]

\[
\begin{array}{c}
L(w) = \{\neg A, \forall r. (\neg B \sqcup A)\}
\end{array}
\]

\[
\begin{array}{c}
L(x) = \{B, \neg B \sqcup A, \neg B, A\}
\end{array}
\]
Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

$$\Delta^\mathcal{I} = \{ u, v, w, x \}$$

$$A^\mathcal{I} = \{ x \}$$

$$B^\mathcal{I} = \{ x \}$$

$$r^\mathcal{I} = \{ \langle u, v \rangle, \langle u, w \rangle, \langle v, x \rangle \}$$

Check that indeed $C^\mathcal{I} = \{ u \}$, given the defined semantics of $\mathcal{ALC}$
Tableau Algorithm Properties

1. the model is finite: only finitely many elements in the domain
2. the model is tree-shaped: the tableau is a labeled tree

the algorithm will always construct finite trees
- from a clash-free tableau, we can construct a finite model
- if there is no clash-free tableau, there is no model
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $ALC$ Concepts
- Correctness and Termination
- Summary
Tableau Properties

- the depth (number of nested quantifiers) decreases in every node
- every node is labeled only with subformulae of $C$
- $C$ has only polynomially many subformulae
- if the output is true, we can build a model out of the constructed tableau
- on the other hand, we can use a model of a satisfiable concept to construct a clash-free tableau for this concept
Tableau Algorithm for $\mathcal{ALC}$ Concepts

**Theorem**

1. the algorithm terminates for every input
2. if the output is $true$, then the input concept is satisfiable
3. if the input concept is satisfiable, then the output is $true$. 

Corollary

Every $\mathcal{ALC}$ concept $C$ has the following properties:

1. finite model property: If $C$ has a model, then it has a finite one.
2. tree model property: If $C$ has a model, then it has a tree-shaped one.
Tableau Algorithm for $\mathcal{ALC}$ Concepts

**Theorem**

1. the algorithm terminates for every input
2. if the output is $true$, then the input concept is satisfiable
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Every $\mathcal{ALC}$ concept $C$ has the following properties:

1. finite model property: If $C$ has a model, then it has a finite one.
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Agenda

- Basic Idea of the Tableau Calculus
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Summary

- we now have a constructive method for building model abstractions
- satisfiable $ALC$ concepts always have a finite model that we can construct
- the algorithm is correct, complete and terminating
- serves as basis for practically implemented algorithms
- next: extension to knowledge bases