Game Theory

- Games in Normal Form
- Equilibria
- Dominance
- Mixed Strategies

Strategies

Game model:

S – set of states
$A_1, \ldots, A_n$ – n sets of actions, one for each player
$l_1, \ldots, l_n$ – where $l_i \subseteq A_i \times S$, the legality relations
$g_1, \ldots, g_n$ – where $g_i \subseteq S \times \mathbb{N}$, the goal relations

A strategy x_i for player i maps every state to a legal move for i

x_i: S → A_i (such that $(x_i(S),S) \in l_i$)

n-Player Games in Normal Form

An *n*-player game in normal form is an n+1-tuple

$$\Gamma = (X_1, \ldots, X_n, u)$$

where X_i is the set of strategies for player i and

$$u = (u_1, \ldots, u_n) \colon \bigtimes_{i=1}^{n} X_i \rightarrow \mathbb{N}^i$$

are the utilities of the players for each n-tuple of strategies.

Roshambo

	Rock	Scissors	Paper
Rock	50 50	0 100	100 0
Scissors	100 0	50 50	0 100
Paper	0 100	100 0	50 50

2-Finger-Morra

	1 Finger	2 Fingers
1 Finger	30 70	90 10
2 Fingers	90 10	0 100

Battle of the Sexes

	Ballgame	Opera
Ballgame	3 4	2 2
Opera	1 1	4 3

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	3 3	4 1
Defect	1 4	2 2

Equilibria

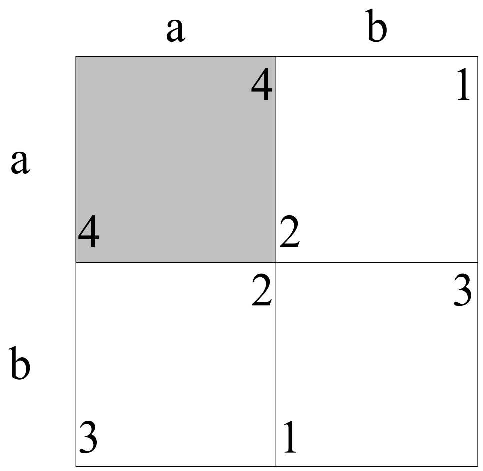
Let $\Gamma = (X_1, \ldots, X_n, u)$ be an n-player game.

$(x_1{}^*, \ldots, x_n{}^*) \in \mathrm{X}_1 \times \ldots \times \mathrm{X}_n$ equilibrium

if for all $i = 1, \ldots, n$ and all $x_i \in \mathrm{X}_i$

$$u_i(x_1{}^*, \ldots, x_{i-1}{}^*, x_i, x_{i+1}{}^*, \ldots, x_n{}^*) \leq u_i(x_1{}^*, \ldots, x_n{}^*)$$

Best Plan

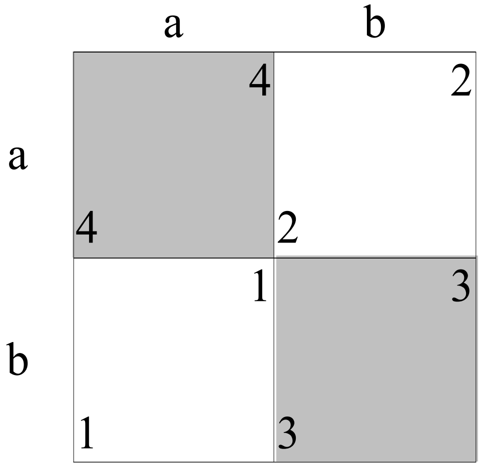


	a	b
a	4 4	1 2
b	2 3	3 1

Battle of the Sexes

	Ballgame	Opera
Ballgame	3 4	2 2
Opera	1 1	4 3

Cooperation



	a	b
a	4 4	2 2
b	1 1	3 3

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	3 3	4 1
Defect	1 4	2 2

Dominance

A strategy $x \in X_i$ dominates a strategy $y \in X_i$ if

$$u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \geq u_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$$

for all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$.

A strategy $x \in X_i$ strongly dominates a strategy $y \in X_i$ if

x dominates y and y does not dominate x.

Assume that opponents are rational: They don't choose a strongly dominated strategy.

Best Plan

	a	b
a	4 4	1 2
~~b~~	2 3	3 1

Iterated Row Dominance

Let a zero-sum game be given by

	a	b	c	d	e
a	10	7	6	9	8
b	10	4	6	9	5
c	9	7	9	8	8
d	2	6	4	3	7

Iterated Row Dominance (2)

	a	b	c	d	e
a	10	7	6	9	8
~~b~~	~~10~~	~~4~~	~~6~~	~~9~~	~~5~~
c	9	7	9	8	8
~~d~~	~~2~~	~~6~~	~~4~~	~~3~~	~~7~~

Iterated Row Dominance (3)

	~~a~~	b	c	~~d~~	~~e~~
a	~~10~~	7	6	~~9~~	~~8~~
c	~~9~~	7	9	~~8~~	~~8~~

Iterated Row Dominance (4)

	b	~~c~~
~~a~~	~~7~~	~~6~~
c	7	~~9~~

Roshambo

	Rock	Scissors	Paper
Rock	50 50	0 100	100 0
Scissors	100 0	50 50	0 100
Paper	0 100	100 0	50 50

This game has no equilibria

2-Finger-Morra

	1 Finger	2 Fingers
1 Finger	30 70	90 10
2 Fingers	90 10	0 100

This game, too, has no equilibria

Mixed Strategies

Let $(X_1, ..., X_n, u)$ be an n-player game, then its mixed extension is

$$\Gamma = (\mathrm{P}_1, ..., \mathrm{P}_n, (e_1, ..., e_n))$$

where for each i=1, ..., n

$$\mathrm{P}_i = \{\mathrm{p}_i\text{: } \mathrm{p}_i \text{ probability measure over } X_i\}$$

and for each $(p_1, ..., p_n) \in \mathrm{P}_1 \times ... \times \mathrm{P}_n$

$$e_i(p_1, ..., p_n) = \sum_{x_1 \in X_1} ... \sum_{x_n \in X_n} u_i(x_1, ..., x_n) * p_1(x_1) * ... * p_n(x_n)$$

Existence of Equilibria

Nash's Theorem.

Every mixed extension of an *n*-player game has at least one equilibrium.

Roshambo

	Rock	Scissors	Paper
Rock	50 50	0 100	100 0
Scissors	100 0	50 50	0 100
Paper	0 100	100 0	50 50

The unique equilibrium is

$$\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$$

2-Finger-Morra

	1 Finger	2 Fingers
1 Finger	30 70	90 10
2 Fingers	90 10	0 100

The unique equilibrium is

$$(p_1^*, p_2^*) = \left(\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{3}{5}, \frac{2}{5}\right)\right)$$

with $e_1(p_1^*, p_2^*) = 46$ and $e_2(p_1^*, p_2^*) = 54$

Iterated Row Dominance for Mixed Strategies

Let a zero-sum game be given by

	a	b	c
a	10	0	8
b	6	4	4
c	3	8	7

Then $p_1 = \left(\frac{1}{2},0,\frac{1}{2}\right)$ dominates $p_1' = (0,1,0)$.

Hence, for all $(p_a', p_b', p_c') \in \mathrm{P}_1$ with $p_b' > 0$ there exists a dominating strategy $(p_a, 0, p_c) \in \mathrm{P}_1$.

Iterated Row Dominance for Mixed Strategies (2)

	a	b	c
a	10	0	8
~~b~~	~~6~~	~~4~~	~~4~~
c	3	8	7

Now $p_2 = \left(\frac{1}{2},\frac{1}{2},0\right)$ dominates $p_2' = (0,0,1)$.

Iterated Row Dominance for Mixed Strategies (3)

	a	b	~~c~~
a	10	0	~~8~~
c	3	8	~~7~~

The unique equilibrium is $\left(\left(\frac{1}{3},0,\frac{2}{3}\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right)$.