Lecture 4

Local Consistency

Outline

- Introduce several local consistency notions:
  - node consistency
  - arc consistency, hyper-arc consistency, directional arc consistency
  - path consistency, directional path consistency
  - *k*-consistency, strong *k*-consistency
  - relational consistency
- Use the proof theoretic framework to characterize these notions

Node Consistency

- CSP is node consistent if for every variable x every unary constraint on x coincides with the domain of x.

- Examples:
  Assume $\mathcal{C}$ contains no unary constraints.

  $\mathbb{N}$ - natural numbers
  $\mathbb{Z}$ - integers
  - $\langle \mathcal{C}, x_1 \geq 0, ..., x_n \geq 0 ; x_1 \in \mathbb{N}, ..., x_n \in \mathbb{N} \rangle$ is node consistent

  - $\langle \mathcal{C}, x_1 \geq 0, ..., x_n \geq 0 ; x_1 \in \mathbb{N}, ..., x_{n-1} \in \mathbb{N}, x_n \in \mathbb{Z} \rangle$ is not node consistent

Arc Consistency

- A constraint C on the variables x, y with the domains X and Y (so $C \subseteq X \times Y$) is arc consistent if
  - $\forall a \in X \exists b \in Y \ (a,b) \in C$
  - $\forall b \in Y \exists a \in X \ (a,b) \in C$

- A CSP is arc consistent if all its binary constraints are

- Examples:
  - $\langle x < y \ ; \ x \in [2..6], \ y \in [3..7] \rangle$ is arc consistent
  - $\langle x < y \ ; \ x \in [2..7], \ y \in [3..7] \rangle$ is not arc consistent

Status of Arc Consistency

- Arc consistency does not imply consistency!

Example: $\langle x = y, x \neq y \,;\, x \in \{a,b\}, y \in \{a,b\}\rangle$

- Consistency does not imply arc consistency!

Example: $\langle x = y \,;\, x \in \{a,b\}, y \in \{a\}\rangle$

- For some CSP's arc consistency does imply consistency.
  (A general result later.)

Proof Rules for Arc Consistency

ARC CONSISTENCY 1

$$\frac{C\,;\,x \in D_x\,,\,y \in D_y}{C\,;\,x \in D'_x\,,\,y \in D_y}$$

where $D'_x := \{a \in D_x \mid \exists b \in D_y\ (a,b) \in C\}$

ARC CONSISTENCY 2

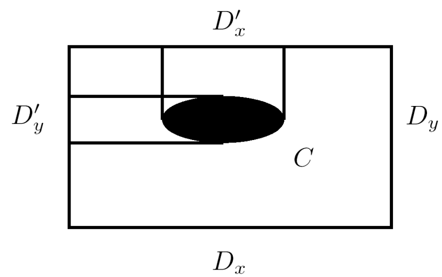
$$\frac{C\,;\,x \in D_x\,,\,y \in D_y}{C\,;\,x \in D_x\,,\,y \in D'_y}$$

where $D'_y := \{b \in D_y \mid \exists a \in D_x\ (a,b) \in C\}$

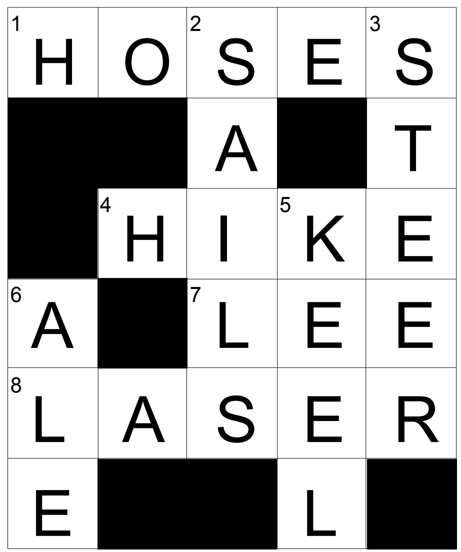
A CSP is arc consistent iff it is closed under the applications of the ARC CONSISTENCY rules 1 and 2.

Intuition and Example

The ARC CONSISTENCY rules

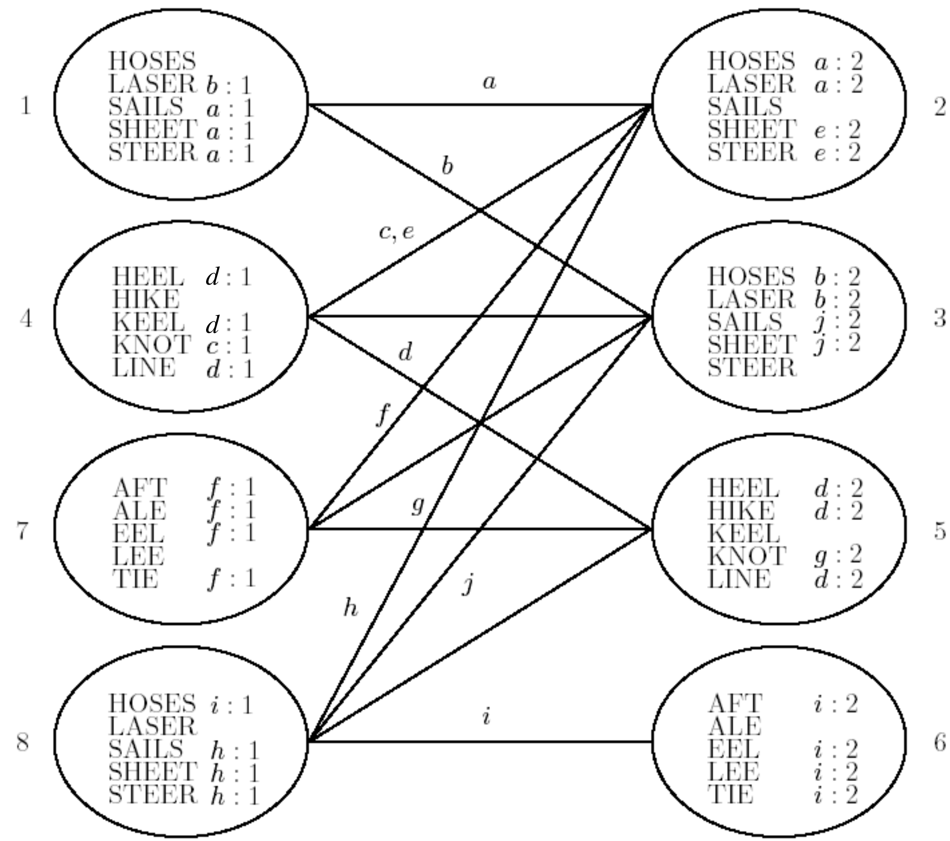


Example



Example, ctd

$a : C_{1,2}$, $b : C_{1,3}$, $c : C_{4,2}$, $d : C_{4,5}$, $e : C_{4,2}$, $f : C_{7,2}$, $g : C_{7,5}$, $h : C_{8,2}$, $i : C_{8,6}$, $j : C_{8,3}$



Hyper-arc Consistency

- A constraint C on the variables $x_1, ..., x_n$ with the domains $D_1, ..., D_n$ is hyper-arc consistent if

$$\forall i \in [1..n] \forall a \in D_i \; \exists d \in C \; a = d[x_i]$$

- CSP is hyper-arc consistent if all its constraints are
- Examples:
  - $\langle x \wedge y = z \,;\, x = 1, y \in \{0,1\}, z \in \{0,1\} \rangle$ is hyper-arc consistent
  - $\langle x \wedge y = z \,;\, x \in \{0,1\}, y \in \{0,1\}, z = 1 \rangle$ is not hyper-arc consistent

Characterization of Hyper-arc Consistency

HYPER-ARC CONSISTENCY

$$\frac{\langle C\,;\,x_1 \in D_1, \ldots, x_n \in D_n \rangle}{\langle C\,;\ldots, x_i \in D'_y, \ldots \rangle}$$

- where C a constraint on the variables $x_1, \ldots, x_n$, $i \in [1..n]$
- $D'_i \coloneqq \{a \in D_i \mid \exists d \in C\ a = d[x_i]\}$

A CSP is hyper-arc consistent iff it is closed under the applications of the HYPER-ARC CONSISTENCY rule.

Directional Arc Consistency

Assume a linear ordering $\prec$ on the variables

- A constraint C on x, y with the domains D_x and D_y is directionally arc consistent w.r.t. $\prec$ if

  - $\forall a \in D_x \, \exists b \in D_y \, (a,b) \in C$, provided $x \prec y$

  - $\forall b \in D_y \, \exists a \in D_x \, (a,b) \in C$, provided $y \prec x$

- A CSP is directionally arc consistent w.r.t. $\prec$ if all its binary constraints are

Example:

$$\langle x < y \, ; \, x \in [2..7], \, y \in [3..7] \rangle$$

- not arc consistent
- directionally arc consistent w.r.t. $y \prec x$
- not directionally arc consistent w.r.t. $x \prec y$

Characterization of Directional Arc Consistency

$\mathcal{P}_{\prec} := \mathcal{P}$ with the variables reordered w.r.t. $\prec$

Example:

Take $\mathcal{P} := \langle x < y,\ y \neq z\ ;\ x \in [2..10],\ y \in [3..7],\ z \in [3..6]\rangle$

and $y \prec x \prec z$

Then $\mathcal{P}_{\prec} := \langle y > x,\ y \neq z\ ;\ y \in [3..7],\ x \in [2..10],\ z \in [3..6]\rangle$

A CSP $\mathcal{P}$ is directionally arc consistent w.r.t. $\prec$ iff the CSP $\mathcal{P}_{\prec}$ is closed under the applications of the ARC CONSISTENCY rule 1.

Limitations of Arc Consistency

Example:

$\langle x < y,\ y < z,\ z < x\ ;\ x, y, z \in [1..100000]\rangle$ is inconsistent

Applying ARC CONSISTENCY rule 1 we get

$\langle x < y,\ y < z,\ z < x\ ;\ x \in [1..99999],\ y, z \in [1..100000]\rangle$ etc

Disadvantages:

- Large number of steps
- Length depends on the size of the domains

Direct proof: use transitivity of <

Path consistency generalizes this form of reasoning to arbitrary binary relations.

Normalized CSP's

A CSP $\mathcal{P}$ is normalized if for each pair *x*, *y* of its variables at most one constraint on *x*, *y* exists.

Denote by $C_{x,y}$ the unique constraint on *x*, *y* if it exists and otherwise the "universal" relation on *x*, *y*.

Consider binary relations *R* and *S*:

- transposition of *R*:

  $R^T := \{(b,a) \mid (a,b) \in R\}$

- composition of *R* and *S*:

  $R \cdot S := \{(a,b) \mid \exists c \ ((a,c) \in R, (c,b) \in S)\}$

Path Consistency

A normalized CSP is path consistent if for each subset $\{x,y,z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}$$

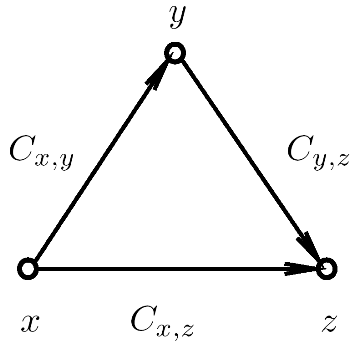
Note: A normalized CSP is path consistent iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C^T_{y,z}$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}$$

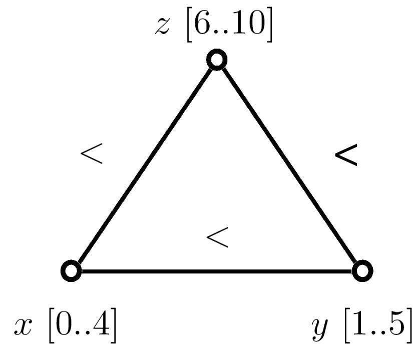
$$C_{y,z} \subseteq C^T_{x,y} \cdot C_{x,z}$$

Intuition:



Path Consistency: Example 1

$\langle x < y,\ y < z,\ x < z\ ;\ x \in [0..4],\ y \in [1..5],\ z \in [6..10]\rangle$ path consistent



$C_{x,y} = \{(a,b) \mid a < b,\ a \in [0..4],\ b \in [1..5]\}$

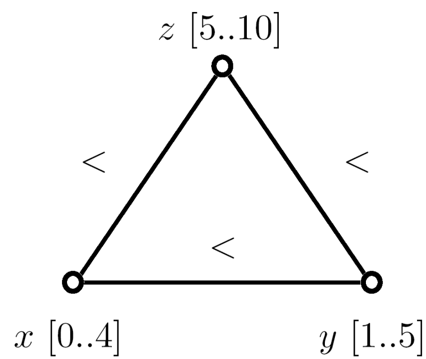
$C_{x,z} = \{(a,c) \mid a < c,\ a \in [0..4],\ c \in [6..10]\}$

$C_{y,z} = \{(b,c) \mid b < c,\ b \in [1..5],\ c \in [6..10]\}$

➸ the 3 conditions (cf. previous slide) are satisfied

Path Consistency: Example 2

$\langle x < y,\ y < z,\ x < z\ ;\ x \in [0..4],\ y \in [1..5],\ z \in [5..10]\rangle$ not path consistent



$C_{x,z} = \{(a,c) \mid a < c,\ a \in [0..4],\ c \in [5..10]\}$

But for $4 \in [0..4]$ and $5 \in [5..10]$ there is no $y \in [1..5]$ s.t. $4 < y$ and $y < 5$.

Characterization of Path Consistency

PATH CONSISTENCY 1

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C'_{x,y}, C_{x,z}, C_{y,z}} \quad \text{where } C'_{x,y} := C_{x,y} \cap C_{x,z} \cdot C^{T}_{y,z}$$

PATH CONSISTENCY 2

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C'_{x,z}, C_{y,z}} \quad \text{where } C'_{x,z} := C_{x,z} \cap C_{x,y} \cdot C_{y,z}$$

PATH CONSISTENCY 3

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C_{x,z}, C'_{y,z}} \quad \text{where } C'_{y,z} := C_{y,z} \cap C^{T}_{x,y} \cdot C_{x,z}$$

A normalized CSP is path consistent iff it is closed under the applications of the PATH CONSISTENCY rules 1, 2, and 3.

m-Path Consistency

A normalized CSP is *m*-path consistent ($m \geq 2$) if for each subset $\{x_1, ..., x_{m+1}\}$ of its variables

$$C_{x_1,x_{m+1}} \subseteq C_{x_1,x_2} \cdot C_{x_2,x_3} \cdot ... \cdot C_{x_m,x_{m+1}}$$

A normalized CSP is m-path consistent if for each subset $\{x_1, ..., x_{m+1}\}$ of its variables

if $(a_1,a_{m+1}) \in C_{x_1,x_{m+1}}$, then for some $a_2, ..., a_m$:

$(a_i,a_{i+1}) \in C_{x_i,x_{i+1}}$ for all $i \in [1..m]$

$a_2, ..., a_m$: path connecting a_1 and a_{m+1}

Theorem

Every normalized, path consistent CSP is m-path consistent for each $m \geq 2$

Proof: Induction on m

Directional Path Consistency

Assume a linear ordering $\prec$ on the variables. A normalized CSP is directionally path consistent w.r.t. $\prec$ if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}, \text{ provided } x, z \prec y$$

A normalized CSP is directionally path consistent w.r.t. $\prec$ iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C^T_{y,z}, \text{ provided } x, y \prec z$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}, \text{ provided } x, z \prec y$$

$$C_{y,z} \subseteq C^T_{x,y} \cdot C_{x,z}, \text{ provided } y, z \prec x$$

Examples

Recall $\langle x < y,\ y < z,\ x < z\ ;\ x \in [0..4],\ y \in [1..5],\ z \in [5..10]\rangle$

$C_{x,y} = \{(a,b) \mid a < b,\ a \in [0..4],\ b \in [1..5]\}$

$C_{x,z} = \{(a,c) \mid a < c,\ a \in [0..4],\ c \in [5..10]\}$

$C_{y,z} = \{(b,c) \mid b < c,\ b \in [1..5],\ c \in [5..10]\}$

- It is directionally path consistent w.r.t. the ordering $\prec$ in which $x, y \prec z$.

  Indeed, for every pair $(a,b) \in C_{x,y}$ there exists $z \in [5..10]$ such that $a < z$ and $b < z$.

- It is directionally path consistent w.r.t. the ordering $\prec$ in which $y, z \prec x$.

  Indeed, for every pair $(b,c) \in C_{y,z}$ there exists $x \in [0..4]$ such that $x < b$ and $x < c$.

- It is not directionally path consistent w.r.t. the ordering $\prec$ in which $x, z \prec y$.

Characterization of Directional Path Consistency

A normalized CSP $\mathcal{P}$ is directionally path consistent w.r.t. $\prec$ iff $\mathcal{P}_{\prec}$ is closed under the applications of the PATH CONSISTENCY rule 1.

Instantiations

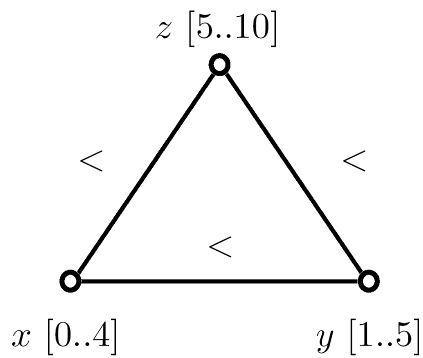
Fix a CSP $\mathcal{P}$.

- Instantiation: function on a subset of the variables of $\mathcal{P}$. It assigns to each variable a value from its domain.
  Notation: $\{(x_1,d_1), ..., (x_k,d_k)\}$
- C: a constraint on $x_1, ..., x_k$
  Instantiation $\{(x_1,d_1), ..., (x_k,d_k)\}$ satisfies C if $(d_1, ..., d_k) \in C$
- I: instantiation with a domain X, $Y \subseteq X$
  $I \mid Y$: restriction of I to Y
- Instantiation I with domain X is consistent if for every constraint C of $\mathcal{P}$ on some Y with $Y \subseteq X$: $I \mid Y$ satisfies C.
- Consistent instantiation is k-consistent if its domain consists of k variables.
- An instantiation is a solution to $\mathcal{P}$ if it is consistent and defined on all variables of $\mathcal{P}$.

Example

Consider $\langle x < y,\ y < z,\ x < z\ ;\ x \in [0..4],\ y \in [1..5],\ z \in [5..10]\rangle$

Let $I \coloneqq \{(x,0), (y,5), (z,6)\}$



- $I \mid \{x,y\} = \{(x,0), (y,5)\}$; it satisfies $x < y$
- $I \mid \{x,z\} = \{(x,0), (z,6)\}$; it satisfies $x < z$
- $I \mid \{y,z\} = \{(y,5), (z,6)\}$; it satisfies $y < z$
- So I is a 3-consistent instantiation. It is a solution to this CSP.

k-Consistency

- CSP is 1-consistent if for every variable *x* with a domain *D* each unary constraint on *x* equals *D*
- CSP is *k*-consistent, $k > 1$, if every $(k-1)$-consistent instantiation can be extended to a *k*-consistent instantiation no matter which new variable is chosen.

k-consistency aka node consistency

Note:

- A node consistent CSP is arc consistent iff it is 2-consistent
- A node consistent, normalized, binary CSP is path consistent iff it is 3-consistent

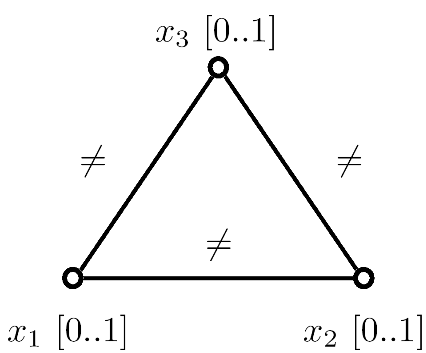
k-Consistency, ctd

Fix k > 1

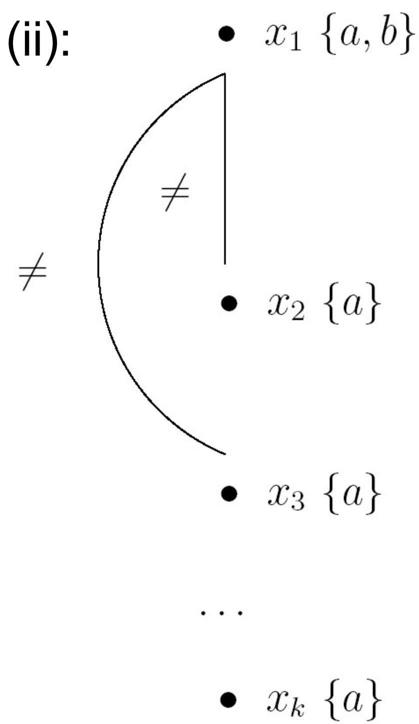
(i) There exists a CSP that is (k – 1)-consistent but not k-consistent

(ii) There exists a CSP that is not (k – 1)-consistent but is k-consistent

Proof of (i) for k = 3:



Proof of (ii):



Strong k-Consistency

CSP strongly k-consistent, $k \geq 1$, if it is i-consistent for every $i \in [1..k]$

Theorem

Take a CSP with k variables, $k \geq 1$, s.t.

- at least one domain is non-empty
- it is strongly k-consistent

Then it is consistent.

Proof: Construct a solution by induction: Prove that

(i) there exists a 1-consistent instantiation

(ii) for every $i \in [2..k]$ each $(i-1)$-consistent instantiation can be extended to an i-consistent instantiation

Disadvantage: Required level of strong consistency = # of variables

Graphs and CSP's

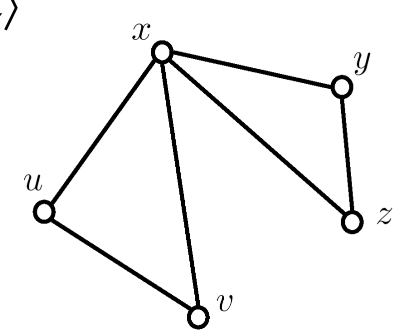
A graph can be associated with a CSP $\mathcal{P}$.

Nodes: variables of $\mathcal{P}$

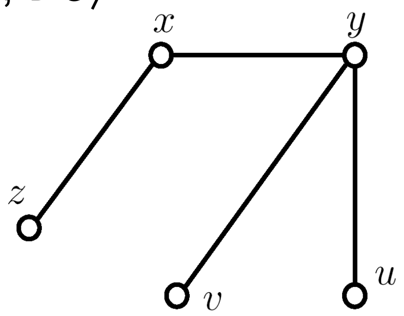
Arcs: connect two variables if they appear jointly in some constraint

Examples

- *SEND* + *MORE* = *MONEY* puzzle
  The graph has 8 nodes, *S*, *E*, *N*, *D*, *M*, *O*, *R*, *Y*, and is complete
- $\langle x + y = z, x + u = v \,;\, \mathcal{DE} \rangle$



- $\langle x < z, x < y, y < u, y < v \,;\, \mathcal{DE} \rangle$



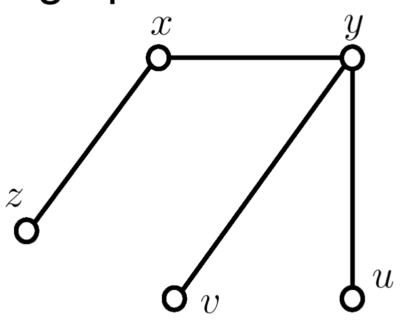
Width of a Graph

G: finite graph

$\prec$: linear ordering on the nodes of *G*

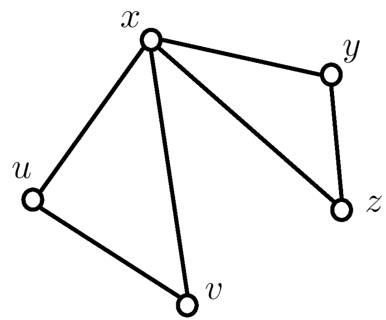
- $\prec$-width of a node of *G*: number of arcs in *G* that connect it to $\prec$-smaller nodes
- $\prec$-width of *G*: maximum of the $\prec$-widths of its nodes
- The width of *G*: minimum of $\prec$-widths for all linear orderings $\prec$

Examples:

- *SEND* + *MORE* = *MONEY* puzzle
  Complete graph with 8 nodes, so its width is 7
- 

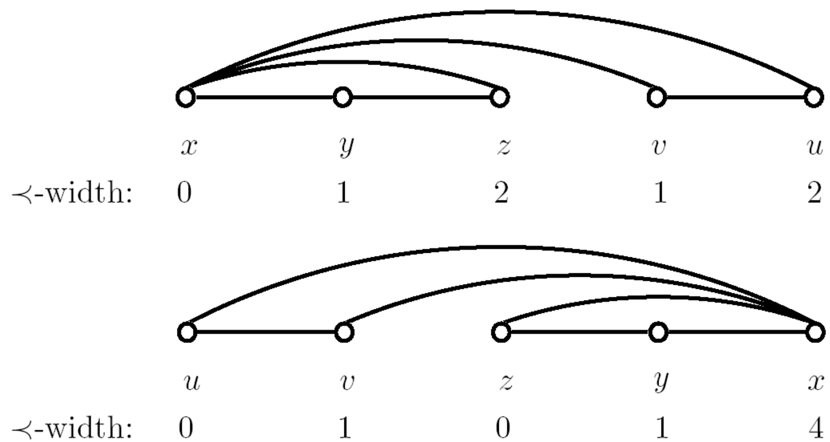
  It is a tree, so its width = 1

Examples, ctd



The width of this graph is 2.

Two examples of the $\prec$-widths of the nodes:



Consistency via Strong k-Consistency

Theorem: Given a CSP such that

- all domains are non-empty
- it is strongly k-consistent
- the graph associated with it has width $k - 1$

Then this CSP is consistent.

Proof: Assume n variables

- Reorder the variables so that the resulting $\prec$-width is $k - 1$
- Prove by induction that
  - there exists consistent instantiation with domain $\{x_1\}$
  - for every $j \in [1..n - 1]$ each consistent instantiation with domain $\{x_1, ..., x_j\}$ can be extended to a consistent instantiation with domain $\{x_1, ..., x_{j+1}\}$

Useful Corollaries

Corollary 1
Given: $\mathcal{P}$ and a linear ordering $\prec$ such that

- all domains are non-empty
- $\mathcal{P}$ is node consistent
- $\mathcal{P}$ is directionally arc consistent w.r.t. $\prec$
- the $\prec$-width of the graph associated with $\mathcal{P}$ is 1

Then $\mathcal{P}$ is consistent.

Corollary 2
Given: $\mathcal{P}$ and a linear ordering $\prec$ such that

- all domains are non-empty
- $\mathcal{P}$ is directionally arc consistent w.r.t. $\prec$
- $\mathcal{P}$ is directionally path consistent w.r.t. $\prec$
- the $\prec$-width of the graph associated with $\mathcal{P}$ is 2

Then $\mathcal{P}$ is consistent.

Relational Consistency

"Ultimate" notion of local consistency

- Given: $\mathcal{P}$ and a subsequence $\mathcal{C}$ of its constraints

  $\mathcal{P} \mid \mathcal{C}$:

  - remove from $\mathcal{P}$ all constraints not in $\mathcal{C}$

  - delete all domain expressions involving variables not present in any constraint $\mathcal{C}$

- $\mathcal{P}$ is relationally (i, m)-consistent if for every sequence $\mathcal{C}$ of m constraints

  and $X \subseteq Var(\mathcal{C})$ of size i:

  every consistent instantiation with the domain X can be extended to a solution to $\mathcal{P} \mid \mathcal{C}$

Intuition:

For every sequence of *m* constraints and for every set *X* of *i* variables, each present in one of these *m* constraints:

Each consistent instantiation with the domain *X* can be extended to a solution to all these *m* constraints.

Relational Consistency, ctd

Some properties:

- A node consistent, binary CSP is arc consistent iff it is relationally (1, 1)-consistent
- A node consistent CSP is hyper-arc consistent iff it is relationally (1, 1)-consistent
- Every node consistent, normalized, relationally (2, 3)-consistent CSP is path consistent
- Every relationally (k – 1, k)-consistent CSP with only binary constraints is k-consistent
- A CSP with m constraints is consistent iff it is relationally (0, m)-consistent

Some Notation

- Given: constraint C on variables X, subsequence Y of X
  $\prod_Y(C) \coloneqq \{d[Y] \mid d \in C\}$

- Given: a sequence of constraints $C_1, ..., C_m$ on variables $X_1, ..., X_m$
  $C_1 \bowtie ... \bowtie C_m \coloneqq \{d \mid d[X_i] \mid \in C_i \text{ for } i \in [1..m]\}$
  $C_1 \bowtie ... \bowtie C_m$ is a constraint on the "union" of $X_1, ..., X_m$

Characterization of Relational Consistency

RELATIONAL (*i, m*)-CONSISTENCY

$$\frac{C_X}{C_X \cap \prod_X (C_1 \bowtie \ldots \bowtie C_m)}$$

If a regular CSP is closed under the applications of RELATIONAL (*i*, *m*)-CONSISTENCY rule for each subsequence of constraints $C_1, \ldots, C_m$ and each subsequence X of $Var(C_1, \ldots, C_m)$ of length i, then it is relationally (*i*, *m*)-consistent.

Objectives

- Introduce several local consistency notions:
  - node consistency
  - arc consistency, hyper-arc consistency, directional arc consistency
  - path consistency, directional path consistency
  - *k*-consistency, strong *k*-consistency
  - relational consistency

- Use the proof theoretic framework to characterize these notions