

Lecture 4

Local Consistency

Outline

- Introduce several local consistency notions:
 - node consistency
 - arc consistency, hyper-arc consistency, directional arc consistency
 - path consistency, directional path consistency
 - k -consistency, strong k -consistency
 - relational consistency
- Use the proof theoretic framework to characterize these notions

Node Consistency

- CSP is **node consistent** if for every variable x every unary constraint on x coincides with the domain of x .
- Examples:
Assume \mathcal{C} contains no unary constraints.

\mathbb{N} - natural numbers

\mathbb{Z} - integers

- $\langle \mathcal{C}, x_1 \geq 0, \dots, x_n \geq 0 ; x_1 \in \mathbb{N}, \dots, x_n \in \mathbb{N} \rangle$ is node consistent
- $\langle \mathcal{C}, x_1 \geq 0, \dots, x_n \geq 0 ; x_1 \in \mathbb{N}, \dots, x_{n-1} \in \mathbb{N}, x_n \in \mathbb{Z} \rangle$ is not node consistent

Arc Consistency

- A constraint C on the variables x, y with the domains X and Y (so $C \subseteq X \times Y$) is **arc consistent** if
 - $\forall a \in X \exists b \in Y (a, b) \in C$
 - $\forall b \in Y \exists a \in X (a, b) \in C$
- A CSP is **arc consistent** if all its binary constraints are
- Examples:
 - $\langle x < y ; x \in [2..6], y \in [3..7] \rangle$ is arc consistent
 - $\langle x < y ; x \in [2..7], y \in [3..7] \rangle$ is not arc consistent

Status of Arc Consistency

- Arc consistency does not imply consistency!

Example: $\langle x = y, x \neq y ; x \in \{a,b\}, y \in \{a,b\} \rangle$

- Consistency does not imply arc consistency!

Example: $\langle x = y ; x \in \{a,b\}, y \in \{a\} \rangle$

- For some CSP's arc consistency does imply consistency.
(A general result later.)

Proof Rules for Arc Consistency

ARC CONSISTENCY 1

$$\frac{C; x \in D_x, y \in D_y}{C; x \in D'_x, y \in D_y}$$

where $D'_x := \{a \in D_x \mid \exists b \in D_y (a,b) \in C\}$

ARC CONSISTENCY 2

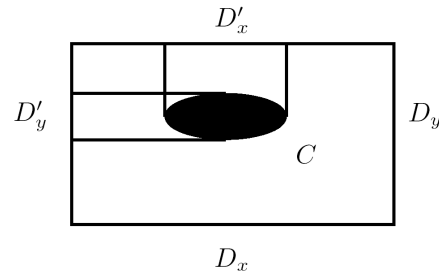
$$\frac{C; x \in D_x, y \in D_y}{C; x \in D_x, y \in D'_y}$$

where $D'_y := \{b \in D_y \mid \exists a \in D_x (a,b) \in C\}$

A CSP is arc consistent iff it is closed under the applications of the ARC CONSISTENCY rules 1 and 2.

Intuition and Example

The ARC CONSISTENCY rules

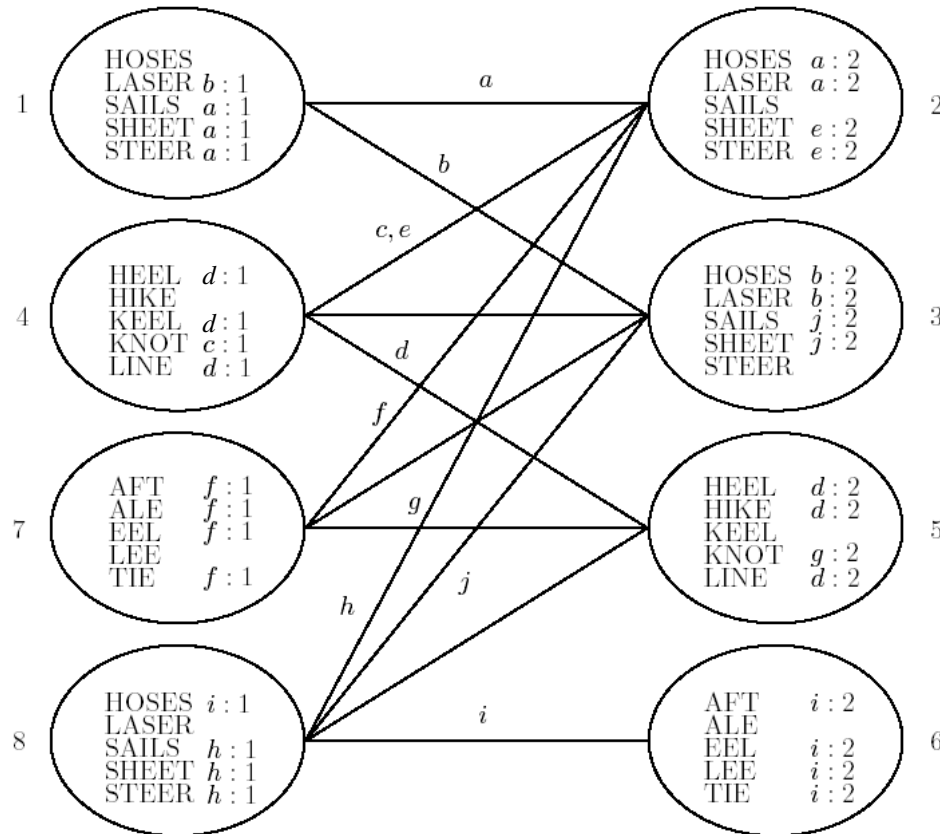


Example

1	H	O	2	S	E	3	S
			A				T
	4	H	I	5	K		E
6	A		7	L	E		E
8	L	A	S	E			R
	E			L			

Example, ctd

$a : C_{1,2}$, $b : C_{1,3}$, $c : C_{4,2}$, $d : C_{4,5}$, $e : C_{4,2}$, $f : C_{7,2}$, $g : C_{7,5}$, $h : C_{8,2}$, $i : C_{8,6}$, $j : C_{8,3}$



Hyper-arc Consistency

- A constraint C on the variables x_1, \dots, x_n with the domains D_1, \dots, D_n is **hyper-arc consistent** if

$$\forall i \in [1..n] \forall a \in D_i \exists d \in C \ a = d[x_i]$$

- CSP is **hyper-arc consistent** if all its constraints are
- Examples:
 - $\langle x \wedge y = z ; x = 1, y \in \{0,1\}, z \in \{0,1\} \rangle$ is hyper-arc consistent
 - $\langle x \wedge y = z ; x \in \{0,1\}, y \in \{0,1\}, z = 1 \rangle$ is not hyper-arc consistent

Characterization of Hyper-arc Consistency

HYPER-ARC CONSISTENCY

$$\frac{\langle C; x_1 \in D_1, \dots, x_n \in D_n \rangle}{\langle C; \dots, x_i \in D'_y, \dots \rangle}$$

- where C a constraint on the variables $x_1, \dots, x_n, i \in [1..n]$
- $D'_i := \{a \in D_i \mid \exists d \in C \ a = d[x_i]\}$

A CSP is hyper-arc consistent iff it is closed under the applications of the HYPER-ARC CONSISTENCY rule.

Directional Arc Consistency

Assume a linear ordering \prec on the variables

- A constraint C on x, y with the domains D_x and D_y is **directionally arc consistent w.r.t. \prec** if
 - $\forall a \in D_x \exists b \in D_y (a, b) \in C$, provided $x \prec y$
 - $\forall b \in D_y \exists a \in D_x (a, b) \in C$, provided $y \prec x$
- A CSP is **directionally arc consistent w.r.t. \prec** if all its binary constraints are

Example:

$$\langle x < y ; x \in [2..7], y \in [3..7] \rangle$$

- not arc consistent
- directionally arc consistent w.r.t. $y \prec x$
- not directionally arc consistent w.r.t. $x \prec y$

Characterization of Directional Arc Consistency

$\mathcal{P}_{\prec} := \mathcal{P}$ with the variables reordered w.r.t. \prec

Example:

Take $\mathcal{P} := \langle x < y, y \neq z ; x \in [2..10], y \in [3..7], z \in [3..6] \rangle$

and $y \prec x \prec z$

Then $\mathcal{P}_{\prec} := \langle y > x, y \neq z ; y \in [3..7], x \in [2..10], z \in [3..6] \rangle$

A CSP \mathcal{P} is directionally arc consistent w.r.t. \prec iff the CSP \mathcal{P}_{\prec} is closed under the applications of the ARC CONSISTENCY rule 1.

Limitations of Arc Consistency

Example:

$\langle x < y, y < z, z < x ; x, y, z \in [1..100000] \rangle$ is inconsistent

Applying ARC CONSISTENCY rule 1 we get

$\langle x < y, y < z, z < x ; x \in [1..99999], y, z \in [1..100000] \rangle$ etc

Disadvantages:

- Large number of steps
- Length depends on the size of the domains

Direct proof: use transitivity of $<$

Path consistency generalizes this form of reasoning to arbitrary binary relations.

Normalized CSP's

A CSP \mathcal{P} is **normalized** if for each pair x, y of its variables at most one constraint on x, y exists.

Denote by $C_{x,y}$ the unique constraint on x, y if it exists and otherwise the “universal” relation on x, y .

Consider binary relations R and S :

- **transposition** of R :

$$R^T := \{(b,a) \mid (a,b) \in R\}$$

- **composition** of R and S :

$$R \cdot S := \{(a,b) \mid \exists c ((a,c) \in R, (c,b) \in S)\}$$

Path Consistency

A normalized CSP is **path consistent** if for each subset $\{x,y,z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}$$

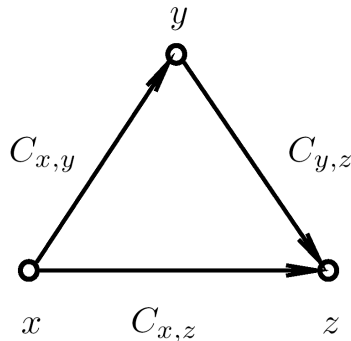
Note: A normalized CSP is path consistent iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C_{y,z}^T$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}$$

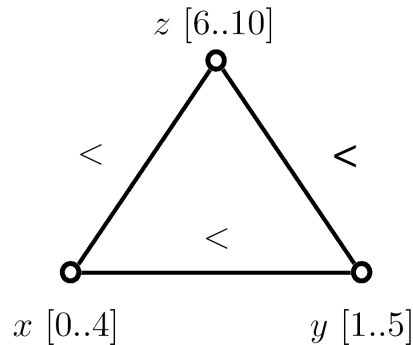
$$C_{y,z} \subseteq C_{x,y}^T \cdot C_{x,z}$$

Intuition:



Path Consistency: Example 1

$\langle x < y, y < z, x < z ; x \in [0..4], y \in [1..5], z \in [6..10] \rangle$ path consistent



$$C_{x,y} = \{(a,b) \mid a < b, a \in [0..4], b \in [1..5]\}$$

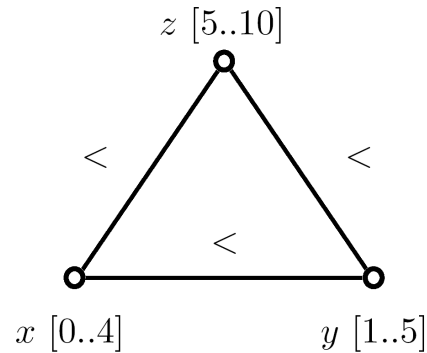
$$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [6..10]\}$$

$$C_{y,z} = \{(b,c) \mid b < c, b \in [1..5], c \in [6..10]\}$$

→ the 3 conditions (cf. previous slide) are satisfied

Path Consistency: Example 2

$\langle x < y, y < z, x < z; x \in [0..4], y \in [1..5], z \in [5..10] \rangle$ not path consistent



$$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [5..10]\}$$

But for $4 \in [0..4]$ and $5 \in [5..10]$ there is no $y \in [1..5]$ s.t. $4 < y$ and $y < 5$.

Characterization of Path Consistency

PATH CONSISTENCY 1

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C'_{x,y}, C_{x,z}, C_{y,z}} \quad \text{where } C'_{x,y} := C_{x,y} \cap C_{x,z} \cdot C^T_{y,z}$$

PATH CONSISTENCY 2

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C'_{x,z}, C_{y,z}} \quad \text{where } C'_{x,z} := C_{x,z} \cap C_{x,y} \cdot C_{y,z}$$

PATH CONSISTENCY 3

$$\frac{C_{x,y}, C_{x,z}, C_{y,z}}{C_{x,y}, C_{x,z}, C'_{y,z}} \quad \text{where } C'_{y,z} := C_{y,z} \cap C^T_{x,y} \cdot C_{x,z}$$

A normalized CSP is path consistent iff it is closed under the applications of the PATH CONSISTENCY rules 1, 2, and 3.

m -Path Consistency

A normalized CSP is **m -path consistent** ($m \geq 2$) if for each subset $\{x_1, \dots, x_{m+1}\}$ of its variables

$$C_{x_1, x_{m+1}} \subseteq C_{x_1, x_2} \cdot C_{x_2, x_3} \cdot \dots \cdot C_{x_m, x_{m+1}}$$

A normalized CSP is m -path consistent if for each subset $\{x_1, \dots, x_{m+1}\}$ of its variables

if $(a_1, a_{m+1}) \in C_{x_1, x_{m+1}}$, then for some a_2, \dots, a_m :

$(a_j, a_{j+1}) \in C_{x_j, x_{j+1}}$ for all $i \in [1..m]$

a_2, \dots, a_m : **path** connecting a_1 and a_{m+1}

Theorem

Every normalized, path consistent CSP is m -path consistent for each $m \geq 2$

Proof: Induction on m

Directional Path Consistency

Assume a linear ordering \prec on the variables. A normalized CSP is **directionally path consistent w.r.t.** \prec if for each subset $\{x, y, z\}$ of its variables

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}, \text{ provided } x, z \prec y$$

A normalized CSP is directionally path consistent w.r.t. \prec iff for each subsequence x, y, z of its variables

$$C_{x,y} \subseteq C_{x,z} \cdot C_{y,z}^T, \text{ provided } x, y \prec z$$

$$C_{x,z} \subseteq C_{x,y} \cdot C_{y,z}, \text{ provided } x, z \prec y$$

$$C_{y,z} \subseteq C_{x,y}^T \cdot C_{x,z}, \text{ provided } y, z \prec x$$

Examples

Recall $\langle x < y, y < z, x < z ; x \in [0..4], y \in [1..5], z \in [5..10] \rangle$

$$C_{x,y} = \{(a,b) \mid a < b, a \in [0..4], b \in [1..5]\}$$

$$C_{x,z} = \{(a,c) \mid a < c, a \in [0..4], c \in [5..10]\}$$

$$C_{y,z} = \{(b,c) \mid b < c, b \in [1..5], c \in [5..10]\}$$

- It is directionally path consistent w.r.t. the ordering \prec in which $x, y \prec z$.
Indeed, for every pair $(a,b) \in C_{x,y}$ there exists $z \in [5..10]$ such that $a < z$ and $b < z$.
- It is directionally path consistent w.r.t. the ordering \prec in which $y, z \prec x$.
Indeed, for every pair $(b,c) \in C_{y,z}$ there exists $x \in [0..4]$ such that $x < b$ and $x < c$.
- It is not directionally path consistent w.r.t. the ordering \prec in which $x, z \prec y$.

Characterization of Directional Path Consistency

A normalized CSP \mathcal{P} is directionally path consistent w.r.t. \prec iff \mathcal{P}_\prec is closed under the applications of the PATH CONSISTENCY rule 1.

Instantiations

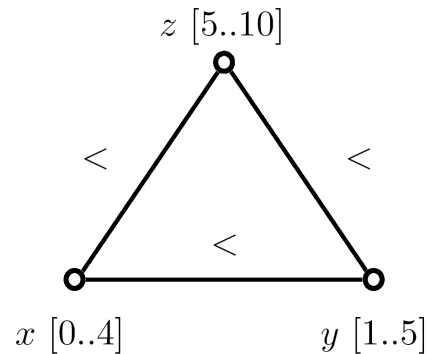
Fix a CSP \mathcal{P} .

- **Instantiation**: function on a subset of the variables of \mathcal{P} . It assigns to each variable a value from its domain.
Notation: $\{(x_1, d_1), \dots, (x_k, d_k)\}$
- C : a constraint on x_1, \dots, x_k
Instantiation $\{(x_1, d_1), \dots, (x_k, d_k)\}$ **satisfies** C if $(d_1, \dots, d_k) \in C$
- I : instantiation with a domain $X, Y \subseteq X$
 $I \upharpoonright Y$: **restriction** of I to Y
- Instantiation I with domain X is **consistent** if for every constraint C of \mathcal{P} on some Y with $Y \subseteq X$: $I \upharpoonright Y$ satisfies C .
- Consistent instantiation is **k -consistent** if its domain consists of k variables.
- An instantiation is a **solution** to \mathcal{P} if it is consistent and defined on all variables of \mathcal{P} .

Example

Consider $\langle x < y, y < z, x < z ; x \in [0..4], y \in [1..5], z \in [5..10] \rangle$

Let $I := \{(x,0), (y,5), (z,6)\}$



- $I \upharpoonright \{x,y\} = \{(x,0), (y,5)\}$; it satisfies $x < y$
- $I \upharpoonright \{x,z\} = \{(x,0), (z,6)\}$; it satisfies $x < z$
- $I \upharpoonright \{y,z\} = \{(y,5), (z,6)\}$; it satisfies $y < z$
- So I is a 3-consistent instantiation. It is a solution to this CSP.

k -Consistency

- CSP is **1-consistent** if for every variable x with a domain D each unary constraint on x equals D
- CSP is **k -consistent**, $k > 1$, if every $(k - 1)$ -consistent instantiation can be extended to a k -consistent instantiation no matter which new variable is chosen.

1-consistency aka **node consistency**

Note:

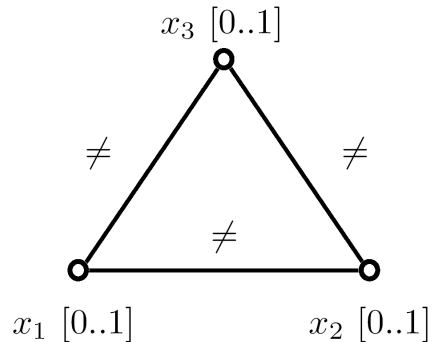
- A node consistent CSP is arc consistent iff it is 2-consistent
- A node consistent, normalized, binary CSP is path consistent iff it is 3-consistent

k -Consistency, ctd

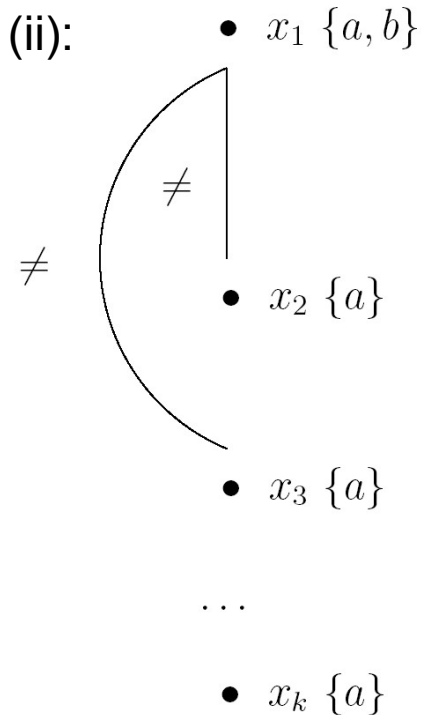
Fix $k > 1$

- (i) There exists a CSP that is $(k - 1)$ -consistent but not k -consistent
- (ii) There exists a CSP that is not $(k - 1)$ -consistent but is k -consistent

Proof of (i) for $k = 3$:



Proof of (ii):



Strong k -Consistency

CSP **strongly k -consistent**, $k \geq 1$, if it is i -consistent for every $i \in [1..k]$

Theorem

Take a CSP with k variables, $k \geq 1$, s.t.

- at least one domain is non-empty
- it is strongly k -consistent

Then it is consistent.

Proof: Construct a solution by induction: Prove that

- there exists a 1-consistent instantiation
- for every $i \in [2..k]$ each $(i - 1)$ -consistent instantiation can be extended to an i -consistent instantiation

Disadvantage: Required level of strong consistency = # of variables

Graphs and CSP's

A graph can be associated with a CSP \mathcal{P} .

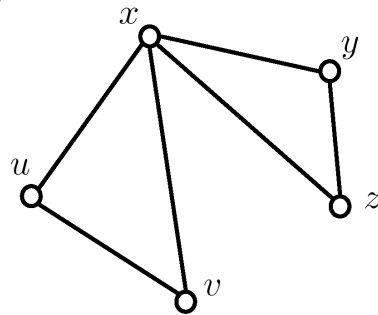
Nodes: variables of \mathcal{P}

Arcs: connect two variables if they appear jointly in some constraint

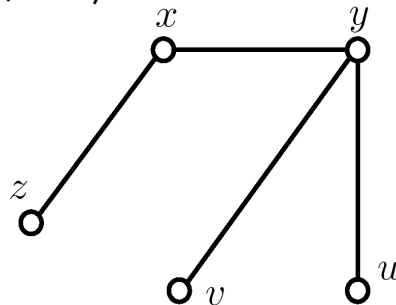
Examples

- $SEND + MORE = MONEY$ puzzle
The graph has 8 nodes, S, E, N, D, M, O, R, Y , and is complete

- $\langle x + y = z, x + u = v; \mathcal{DE} \rangle$



- $\langle x < z, x < y, y < u, y < v; \mathcal{DE} \rangle$



Width of a Graph

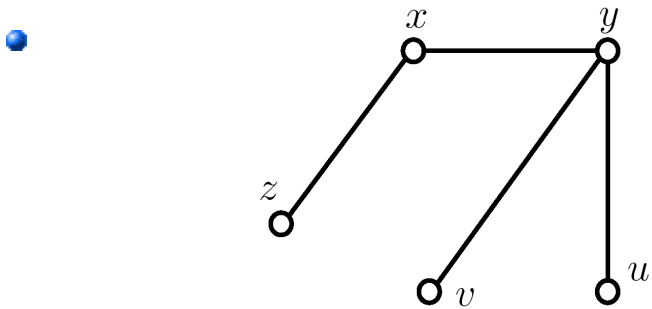
G : finite graph

\prec : linear ordering on the nodes of G

- \prec -width of a node of G : number of arcs in G that connect it to \prec -smaller nodes
- \prec -width of G : maximum of the \prec -widths of its nodes
- The width of G : minimum of \prec -widths for all linear orderings \prec

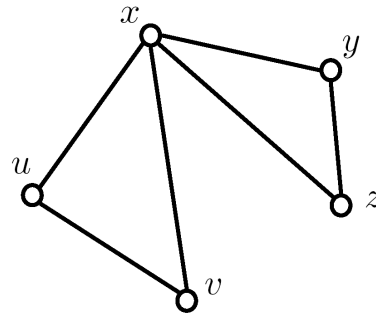
Examples:

- $SEND + MORE = MONEY$ puzzle
Complete graph with 8 nodes, so its width is 7



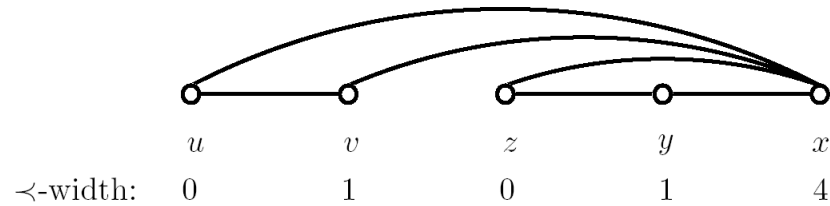
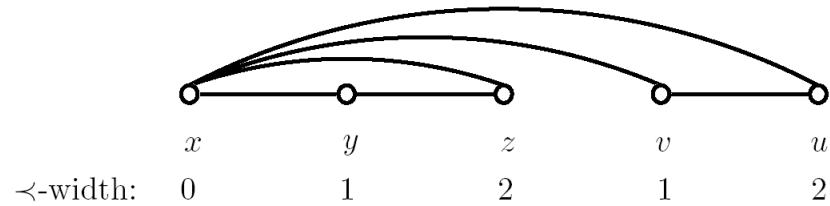
It is a tree, so its width = 1

Examples, ctd



The width of this graph is 2.

Two examples of the \prec -widths of the nodes:



Consistency via Strong k -Consistency

Theorem: Given a CSP such that

- all domains are non-empty
- it is strongly k -consistent
- the graph associated with it has width $k - 1$

Then this CSP is consistent.

Proof: Assume n variables

- Reorder the variables so that the resulting \prec -width is $k - 1$
- Prove by induction that
 - there exists consistent instantiation with domain $\{x_1\}$
 - for every $j \in [1..n - 1]$ each consistent instantiation with domain $\{x_1, \dots, x_j\}$ can be extended to a consistent instantiation with domain $\{x_1, \dots, x_{j+1}\}$

Useful Corollaries

Corollary 1

Given: \mathcal{P} and a linear ordering \prec such that

- all domains are non-empty
- \mathcal{P} is node consistent
- \mathcal{P} is directionally arc consistent w.r.t. \prec
- the \prec -width of the graph associated with \mathcal{P} is 1

Then \mathcal{P} is consistent.

Corollary 2

Given: \mathcal{P} and a linear ordering \prec such that

- all domains are non-empty
- \mathcal{P} is directionally arc consistent w.r.t. \prec
- \mathcal{P} is directionally path consistent w.r.t. \prec
- the \prec -width of the graph associated with \mathcal{P} is 2

Then \mathcal{P} is consistent.

Relational Consistency

“Ultimate” notion of local consistency

- Given: \mathcal{P} and a subsequence \mathcal{C} of its constraints

$\mathcal{P} | \mathcal{C}$:

- remove from \mathcal{P} all constraints not in \mathcal{C}
- delete all domain expressions involving variables not present in any constraint \mathcal{C}

- \mathcal{P} is **relationally (i, m) -consistent** if for every sequence \mathcal{C} of m constraints

and $X \subseteq \text{Var}(\mathcal{C})$ of size i :

every consistent instantiation with the domain X can be extended to a solution to $\mathcal{P} | \mathcal{C}$

Intuition:

For every sequence of m constraints and for every set X of i variables, each present in one of these m constraints:

Each consistent instantiation with the domain X can be extended to a solution to all these m constraints.

Relational Consistency, ctd

Some properties:

- A node consistent, binary CSP is arc consistent iff it is relationally $(1, 1)$ -consistent
- A node consistent CSP is hyper-arc consistent iff it is relationally $(1, 1)$ -consistent
- Every node consistent, normalized, relationally $(2, 3)$ -consistent CSP is path consistent
- Every relationally $(k - 1, k)$ -consistent CSP with only binary constraints is k -consistent
- A CSP with m constraints is consistent iff it is relationally $(0, m)$ -consistent

Some Notation

- Given: constraint C on variables X , subsequence Y of X
 $\Pi_Y(C) := \{d[Y] \mid d \in C\}$
- Given: a sequence of constraints C_1, \dots, C_m on variables X_1, \dots, X_m
 $C_1 \bowtie \dots \bowtie C_m := \{d \mid d[X_i] \in C_i \text{ for } i \in [1..m]\}$
 $C_1 \bowtie \dots \bowtie C_m$ is a constraint on the “union” of X_1, \dots, X_m

Characterization of Relational Consistency

RELATIONAL (i, m) -CONSISTENCY

$$\frac{C_x}{C_x \cap \prod_X (C_1 \text{ r } \dots \text{ r } C_m)}$$

If a regular CSP is closed under the applications of RELATIONAL (i, m) -CONSISTENCY rule for each subsequence of constraints C_1, \dots, C_m and each subsequence X of $Var(C_1, \dots, C_m)$ of length i , then it is relationally (i, m) -consistent.

Objectives

- Introduce several local consistency notions:
 - node consistency
 - arc consistency, hyper-arc consistency, directional arc consistency
 - path consistency, directional path consistency
 - k -consistency, strong k -consistency
 - relational consistency
- Use the proof theoretic framework to characterize these notions