Infinite trees over ranked alphabets

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It is easy to see that terms over a ranked alphabet can be represented by trees.

To reason about behavior of nonterminating systems like databases or operating systems one can use infinite trees. But, there is no immediate generalization of terms to infinite terms. However, there is a direct generalization of finite trees over a ranked alphabet to infinite trees over a ranked alphabet.

Let $\mathbb{N}_+$ denote the set of positive natural numbers and let

$$F = \sum_{k \in \mathbb{N}} F^{(k)}$$

be a ranked alphabet, where $f \in F^{(k)}$ represents a function symbol of arity k (or of rank k).

Infinite trees will be represented by partial mappings $t : \mathbb{N}_+^* \twoheadrightarrow F$. For a partial mapping $h : A \twoheadrightarrow B$ the domain of definition will be denoted by $Dom(h) \subseteq A$.

Definition: A tree over a ranked alphabet $F = \sum_{k \in \mathbb{N}} F^{(k)}$ is a partial mapping $t : \mathbb{N}_+^* \twoheadrightarrow F$ such that $Dom(t)$ satisfies the following conditions:

1. $Dom(t)$ is prefix–closed, i.e. if $\alpha, \beta \in \mathbb{N}_+^*, \alpha\beta \in Dom(t)$ then $\alpha \in Dom(t)$.
2. $Dom(t) \neq \emptyset$
3. If $\alpha \in \mathbb{N}_+^*, i, j \in \mathbb{N}, 1 \leq i \leq j$ and $\alpha j \in Dom(t)$ then $\alpha i \in Dom(t)$.
4. If $t(\alpha) = f \in F^{(k)}$, then $\alpha i \in Dom(t)$ if and only if $1 \leq i \leq k$.

The sequences $\alpha \in Dom(t) \subseteq \mathbb{N}_+^*$ are called the nodes of the tree.

A tree t is called finite if $Dom(t)$ is finite and respectively infinite if $Dom(t)$ is infinite.

Definition: Let $t_1, t_2 : \mathbb{N}_+^* \twoheadrightarrow F$ be trees over F. $t_2 : \mathbb{N}_+^* \twoheadrightarrow F$ is called a subtree of $t_1 : \mathbb{N}_+^* \twoheadrightarrow F$, if there is a node $\alpha \in Dom(t_1)$ such that

1. $\alpha\beta \in Dom(t_1)$ if and only if $\beta \in Dom(t_2)$.
2. If $\beta \in Dom(t_2)$ then $t_2(\beta) = t_1(\alpha\beta)$.

Let $t : \mathbb{N}_+^* \twoheadrightarrow F$ be a tree and α a node of the tree, then t/α denotes the subtree of t with $Dom(t/\alpha) = \{\beta \in \mathbb{N}_+^* | \alpha\beta \in Dom(t)\}$ and $(t/\alpha)(\beta) = t(\alpha\beta)$ for all $\beta \in Dom(t/\alpha)$.

Let F_1 be the ranked alphabet with $F^{(k)} = \emptyset$ for all $k \neq 1$ and $F_1^{(1)} = \{f\}$. How many infinite trees over F_1 exist? How many subtrees exists for a given infinite tree over F_1 ?

Answer the same questions for the ranked alphabet F_2 with $F^{(k)} = \emptyset$ for all $k \neq 2$ and $F_2^{(2)} = \{g\}$.

An infinite tree over a ranked alphabet is called regular if the tree has only finitely many subtrees.

Give an example of a tree which is not regular.

Let $F = \sum_{k \in \mathbb{N}} F^{(k)}$ be a ranked alphabet and V a countable set of variables (with $F^{(0)} \cap V = \emptyset$).

$F \cup V$ denotes the ranked alphabet with $(F \cup V)^{(k)} = F^{(k)}$ for all $k \geq 1$ and $(F \cup V)^{(0)} = F^{(0)} \cup V$.

Trees over $F \cup V$ can also be understood as finite and infinite terms over F with variables out of V.

$TU^{\infty}_{F,V}$ denotes the infinitary term universe over F and V.

In the following we extend the substitution of variables from finite terms (trees) to infinite terms (trees).

Let $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ be a substitution and $Dom(\theta) = \{x_1, \ldots, x_n\}$ its domain of definition. Beside $\{t_1, \ldots, t_n\} \subseteq TU^{\infty}_{F,V}$ let be $t \in TU^{\infty}_{F,V}$.

The result $t\theta$ of substituting the variables $\{x_1, \ldots, x_n\}$ in t according to $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ is defined as follows:

1. $Dom(t\theta) = (Dom(t) \setminus t^{-1}(Dom(\theta))) \cup \bigcup_{x_i \in Dom(\theta)} Dom(t\theta)_{x_i/t_i}$

   where $Dom(t\theta)_{x_i/t_i} = \bigcup_{\alpha \in t^{-1}(x_i)} \{\alpha\beta \mid \beta \in Dom(t_i)\}$.

2. If $\beta \in Dom(t) \setminus t^{-1}(Dom(\theta))$ then $(t\theta)(\beta) = t(\beta)$.

   If $\alpha \in t^{-1}(x_i)$ and $\beta \in Dom(t_i)$ then $(t\theta)(\alpha\beta) = t_i(\beta)$ for $\theta(x_i) = t_i$ and $i = 1, \ldots, n$.