Chapter 4

Declarative Interpretation

Outline

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models

What is an Interpretation?

```
direct(frankfurt,san_francisco).
direct(frankfurt,chicago).
direct(san_francisco,honolulu).
direct(honolulu,maui).

connection(X, Y) :- direct(X, Y).
connection(X, Y) :- direct(X, Z), connection(Z, Y).
```

$D = \{FRA, DRS, ORD, SFO, \ldots\}$

$\texttt{frankfurt}_J = FRA$, $\texttt{chicago}_J = ORD$, $\texttt{san\_francisco}_J = SFO$, ...

$\texttt{direct}_I = \{(FRA, SFO), (FRA, ORD), \ldots\}$

$\texttt{connection}_I = \{(FRA, SFO), (FRA, ORD), (FRA, HNL), \ldots\}$

What is an Interpretation?

```
add(X,0,X).
add(X,s(Y),s(Z)) :- add(X,Y,Z).
```

$D = \mathbb{N}$

$\mathtt{0}_J = 0$

$\mathtt{s}_J : \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathtt{s}_J(n) = n + 1$

$\mathtt{add}_I = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), ...\}$

Another Example

```
add(X,0,X).
add(X,s(Y),s(Z)) :- add(X,Y,Z).
```

$D = \{0, s(0), s(s(0)), \ldots\}$

$\mathtt{0}_J = 0$

$\mathtt{s}_J : D \rightarrow D$ such that $\mathtt{s}_J(t) = s(t)$

$\mathtt{add}_I = \{(0, 0, 0), (s(0), 0, s(0)), (0, s(0), s(0)), (s(0), s(0), s(s(0))), \ldots\}$

(This will be called a “Herbrand model”.)

Algebras

V set of variables, F ranked alphabet of function symbols: An algebra J for F (or pre-interpretation for F) consists of:

1. domain $:\Leftrightarrow$ non-empty set D
2. assignment of a mapping

$$f_J : D^n \to D$$

   to every $f \in F^{(n)}$ with $n \geq 0$

State σ over D $\quad :\Leftrightarrow \quad$ mapping $\sigma : V \to D$

Extension of σ to $TU_{F,V}$ $\quad :\Leftrightarrow \quad$ $\sigma : TU_{F,V} \to D$ such that for every $f \in F^{(n)}$

$$\sigma(f(t_1, \ldots, t_n)) = f_J(\sigma(t_1), \ldots, \sigma(t_n))$$

Interpretations

F ranked alphabet of function symbols, $\prod$ ranked alphabet of predicate symbols:

An interpretation I for F and $\prod$ consists of:

1. algebra J for F (with domain D)
2. assignment of a relation

$$p_I \subseteq \underbrace{D \times \ldots \times D}_{n}$$

to every $p \in \prod^{(n)}$ with $n \geq 0$

Herbrand Universes and Bases

Recall $TU_{F,V}$:⇔ term universe over function symbols F, variables V

$TB_{\prod,F,V}$:⇔ term base (i.e., all atoms) over predicate symbols $\prod$ and F, V

- Herbrand universe $HU_F :\Leftrightarrow TU_{F,\emptyset}$
- Herbrand base $HB_{\prod,F} :\Leftrightarrow TB_{\prod,F,\emptyset}$

Interpretations (Example)

Let P_{add} "add-program".

I_1, I_2, I_3, I_4, I_5, and I_6 are interpretations for $\{s, 0\}$ and $\{add\}$:

I_1: $D_{I_1} = \mathbb{N}$, $0_{I_1} = 0$, $s_{I_1}(n) = n + 1$ for each $n \in \mathbb{N}$, $add_{I_1} = \{(m, n, m + n) \mid m, n \in \mathbb{N}\}$

I_2: $D_{I_2} = \mathbb{N}$, $0_{I_2} = 0$, $s_{I_2}(n) = n + 1$ for each $n \in \mathbb{N}$, $add_{I_2} = \{(m, n, m * n) \mid m, n \in \mathbb{N}\}$

I_3: $D_{I_3} = HU_{\{s,\, 0\}}$, $0_{I_3} = 0$, $s_{I_3}(t) = s(t)$ for each $t \in HU_{\{s,\, 0\}}$,

$add_{I_3} = \{(s^m(0), s^n(0), s^{m+n}(0)) \mid m, n \in \mathbb{N}\}$

I_4: $D_{I_4} = HU_{\{s,\, 0\}}$, $0_{I_4} = 0$, $s_{I_4}(t) = s(t)$ for each $t \in HU_{\{s,\, 0\}}$, $add_{I_4} = \emptyset$

I_5: $D_{I_5} = HU_{\{s,\, 0\}}$, $0_{I_5} = 0$, $s_{I_5}(t) = s(t)$ for each $t \in HU_{\{s,\, 0\}}$, $add_{I_5} = (HU_{\{s,\, 0\}})^3$

I_6: $D_{I_6} = \{0, 1\}$, $0_{I_6} = 0$, $s_{I_6}(n) = n$ for each $n \in \{0, 1\}$, $add_{I_6} = \{(m, n, m) \mid m, n \in \{0, 1\}\}$

Logical Truth (I)

E expression $:\Leftrightarrow$ E atom, query, clause, or resultant

E expression, I interpretation, σ state:

E true in I under σ, written: $I \models_\sigma E$

$:\Leftrightarrow$

by case analysis on E:

- $I \models_\sigma p(t_1, \ldots, t_n) :\Leftrightarrow (\sigma(t_1), \ldots, \sigma(t_n)) \in p_I$
- $I \models_\sigma A_1, \ldots, A_n :\Leftrightarrow I \models_\sigma A_i$ for every $i = 1, \ldots, n$
- $I \models_\sigma A \leftarrow \underline{\underline{B}} :\Leftrightarrow$ if $I \models_\sigma \underline{\underline{B}}$ *then* $I \models_\sigma A$
- $I \models_\sigma \underline{\underline{A}} \leftarrow \underline{\underline{B}} :\Leftrightarrow$ if $I \models_\sigma \underline{\underline{B}}$ *then* $I \models_\sigma \underline{\underline{A}}$

Logical Truth (II)

E expression, I interpretation:

Let $x_1, \ldots, x_k$ be the variables occuring in E.

- $\forall x_1, \ldots, \forall x_k\, E$ universal closure of E (abbreviated $\forall E$)
- $\exists x_1, \ldots, \exists x_k\, E$ existential closure of E (abbreviated $\exists E$)
- $I \models \forall E :\Leftrightarrow I \models_\sigma E$ for every state σ
- $I \models \exists E :\Leftrightarrow I \models_\sigma E$ for some state σ
- E true in I (or: I model of E), written: $I \models E :\Leftrightarrow I \models \forall E$

Logical Truth (III)

S, T sets of expressions, I interpretation:

- I model of S, written: $I \models S :\Leftrightarrow I \models E$ for every $E \in S$
- T semantic (or: logical) consequence of S, written $S \models T$
  $:\Leftrightarrow$ every model of S is a model of T

P program, Q_0 query, θ substitution:

- $\theta \mid_{Var(Q_0)}$ correct answer substitution of $Q_0 :\Leftrightarrow P \models Q_0\theta$
- $Q_0\theta$ correct instance of $Q_0 :\Leftrightarrow P \models Q_0\theta$

Models (Example)

Let P_{add} "add-program" and let I_1, I_2, I_3, I_4, I_5, and I_6 be the interpretations from slide 9.

- $I_1 \models P_{add}$ (since $I_1 \models_\sigma c$ for every clause $c \in P_{add}$ and state $\sigma : V \to \mathbb{N}$:
  (i) $(\sigma(x), \sigma(0), \sigma(x)) \in add_{I_1}$ and
  (ii) if $(\sigma(x), \sigma(y), \sigma(z)) \in add_{I_1}$ then $(\sigma(x), \sigma(y)+1, \sigma(z)+1) \in add_{I_1}$)
- $I_2 \not\models P_{add}$ (e.g. let $\sigma(x) = 1$, then $I_2 \not\models_\sigma add(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (1, 0, 1) \notin add_{I_2}$)
- $I_3 \models P_{add}$ (like for I_1; we call I_3 a (least) Herbrand model)
- $I_4 \not\models P_{add}$ (e.g. let $\sigma(x) = s(0)$, then $I_4 \not\models_\sigma add(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (s(0), 0, s(0)) \notin add_{I_4}$)
- $I_5 \models P_{add}$ (like for I_1; we call I_5 a Herbrand model)
- $I_6 \models P_{add}$ (like for I_1)

Semantic Consequences (Example)

Let P_{add} “add-program”.

- $P_{add} \models add(x, 0, x)$
  (for every interpretation I : if $I \models P_{add}$ then $I \models add(x, 0, x)$, since $add(x, 0, x) \in P_{add}$)

- $P_{add} \models add(x, s(0), s(x))$
  (for every interpretation I : if $I \models P_{add}$ then $I \models add(x, 0, x)$
  and $I \models add(x, s(0), s(x)) \leftarrow add(x, 0, x)$ (instance of clause), thus $I \models add(x, s(0), s(x))$)

- $P_{add} \not\models add(0, x, x)$
  (consider interpretation I_6 from slide 9 with $I_6 \models P_{add}$;
  $I_6 \not\models add(0, x, x)$, since e.g. $I_6 \not\models_\sigma add(0, x, x)$ for $\sigma(x) = 1$,
  since $(\sigma(0), \sigma(x), \sigma(x)) = (0, 1, 1) \notin add_{I_6}$)

Towards Soundness of SLD-Resolution (I)

Lemma 4.3 (i)

Let $Q \overset{\theta}{\underset{c}{\Longrightarrow}} Q'$ be an SLD-derivation step and $Q\theta \leftarrow Q'$ the resultant associated with it.

Then $\quad c \models Q\theta \leftarrow Q'$

Proof.

Let $Q = \underline{\underline{A}}, B, \underline{\underline{C}}$ with selected atom B. Let $H \leftarrow \underline{\underline{B}}$ be the input clause and Q' = $(\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}})\theta$.

Then

	$c \models H \leftarrow \underline{\underline{B}}$	(variant of c)
implies	$c \models H\theta \leftarrow \underline{\underline{B}}\theta$	(instance)
implies	$c \models B\theta \leftarrow \underline{\underline{B}}\theta$	(θ unifier)
implies	$c \models (\underline{\underline{A}}, B, \underline{\underline{C}})\theta \leftarrow (\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}})\theta$	(“context” unchanged)

Towards Soundness of SLD-Resolution (II)

Lemma 4.3 (ii)

Let ξ be an SLD-derivation of $P \cup \{Q_0\}$. For $i \geq 0$ let R_i be the resultant of level i of ξ.

Then $\quad P \models R_i$

Proof.

Let $\xi = Q_0 \stackrel{\theta_1}{\Longrightarrow} Q_1 \ldots Q_n \stackrel{\theta_{n+1}}{\Longrightarrow} Q_{n+1} \ldots$ Induction on $i \geq 0$:

$i = 0$: $\quad R_0 = Q_0 \leftarrow Q_0$ = "true", thus $P \models R_0$

$i = 1$: $\quad R_1 = Q_0\theta_1 \leftarrow Q_1$; by Lemma 4.3 (i): $P \models R_1$

$i \rightsquigarrow i + 1$: $\quad R_{i+1} = Q_0\theta_1 \ldots \theta_{i+1} \leftarrow Q_{i+1}$ is a semantic consequence of resultant $Q_i\theta_{i+1} \leftarrow Q_{i+1}$ associated with $(i + 1)$-st derivation step and $R_i\theta_{i+1} = Q_0\theta_1 \ldots \theta_{i+1} \leftarrow Q_i\theta_{i+1}$, thus by Lemma 4.3 (i) and induction hypothesis: $P \models R_{i+1}$

Soundness of SLD-Resolution

Theorem 4.4

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$ with CAS θ, then $P \models Q_0\theta$.

Proof.

Let $\xi = Q_0 \overset{\theta_1}{\Longrightarrow} \ldots \overset{\theta_n}{\Longrightarrow} \Box$ be successful SLD-derivation.

Lemma 4.3 (ii) applied to the resultant of level n of ξ implies $P \models Q_0\theta_1 \ldots \theta_n$ and

$Q_0\theta_1 \ldots \theta_n = Q_0(\theta_1 \ldots \theta_n|_{Var(Q_0)}) = Q_0\theta$.

Comparison to Intuitive Meaning of Queries

Corollary 4.5

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$, then $P \models \exists Q_0$.

Proof.

Theorem 4.4 implies $P \models Q_0\theta$ for some CAS θ.

Then, $P \models Q_0\theta$

implies for every interpretation I: if $I \models P$, then $I \models Q_0\theta$

implies for every interpretation I: if $I \models P$, then $I \models \forall(Q_0\theta)$

implies for every interpretation I: if $I \models P$, then $I \models \exists Q_0$

implies $P \models \exists Q_0$

Towards Completeness of SLD-Resolution

To show completeness of SLD-resolution we need to syntactically characterize the set of semantically derivable queries.

The concepts of term models and implication trees serve this purpose.

Term Models

V set of variables, F function symbols, $\prod$ predicate symbols:

The term algebra J for F is defined as follows:

1. domain $D = TU_{F,V}$
2. mapping $f_J : (TU_{F,V})^n \rightarrow TU_{F,V}$ assigned to every $f \in F^{(n)}$ with
$$f_J(t_1, \ldots, t_n) \Leftrightarrow f(t_1, \ldots, t_n)$$

A term interpretation I for F and $\prod$ consists of:

1. term algebra for F
2. $I \subseteq TB_{\prod,F,V}$ (set of atoms that are true; equivalent: assignment of a relation $p_I \subseteq (TU_{F,V})^n$ to every $p \in \prod^{(n)}$)

I term model of a set S of expressions $:\Leftrightarrow$ I term interpretation and model of S

Herbrand Models

The Herbrand algebra J for F is defined as follows:

1. domain $D = HU_F$
2. mapping $f_J : (HU_F)^n \rightarrow HU_F$ assigned to every $f \in F^{(n)}$ with

$$f_J(t_1, ..., t_n) \Leftrightarrow f(t_1, ..., t_n)$$

A Herbrand interpretation I for F and $\prod$ consists of:

1. Herbrand algebra for F
2. $I \subseteq HB_{\prod,F}$ (set of ground atoms that are true)

I Herbrand model of a set S of expressions :$\Leftrightarrow$ I Herbrand interpretation and model of S

I least Herbrand model of a set S of expressions

:$\Leftrightarrow$ I Herbrand model of S and $I \subseteq I'$ for all Herbrand models I' of S

Implication Trees

implication tree w.r.t. program P

$:\Leftrightarrow$

- finite tree whose nodes are atoms
- if A is a node with the direct descendants $B_1, \ldots, B_n$ then $A \leftarrow B_1, \ldots, B_n \in inst(P)$
- if A is a leaf, then $A \leftarrow\ \in inst(P)$

E expression, S set of expressions:

- $inst(E)$ $:\Leftrightarrow$ set of all instances of E
- $inst(S)$ $:\Leftrightarrow$ set of all instances of Elements $E \in S$
- $ground(E)$ $:\Leftrightarrow$ set of all ground instances of E
- $ground(S)$ $:\Leftrightarrow$ set of all ground instances of Elements $E \in S$

Implication Trees (Example)

Let P_{add} "add-program", $n \in \mathbb{N}$, V set of variables, $t \in TU_{\{s,0\},V}$, and

$$
\mathcal{T} = \quad
\begin{array}{c}
add(t,\ s^n(0),\ s^n(t)) \\
| \\
add(t,\ s^{n-1}(0),\ s^{n-1}(t)) \\
: \\
: \\
add(t,\ s(0),\ s(t)) \\
| \\
add(t,\ 0,\ t)
\end{array}
$$

If $t \in HU_{\{s,0\}}$, then $\mathcal{T}$ is ground implication tree w.r.t. P_{add} .

Implication Trees Constitute Term Model

Lemma 4.7

Consider term interpretation I, atom A, program P

- $I \models A$ iff $inst(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in inst(P)$: if $\{B_1, \ldots, B_n\} \subseteq I$ then $A \in I$

Lemma 4.12

The term interpretation

$\mathcal{C}(P) :\Leftrightarrow \{A \mid A$ is the root of some implication tree w.r.t. $P\}$ is a model of P.

Ground Implication Trees Constitute Herbrand Model

Lemma 4.26

Consider Herbrand interpretation I, atom A, program P

- $I \models A$ iff $ground(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, ..., B_n \in ground(P)$, $\{B_1, ..., B_n\} \subseteq I$ implies $A \in I$

Lemma 4.28

The Herbrand interpretation

$\mathcal{M}(P) :\Leftrightarrow \{A \mid A$ is the root of some ground implication tree w.r.t. $P\}$ is a model of P.

Example

Let P_{add} “add-program”, and V set of variables.

The term interpretation

$\mathcal{C}(P_{add}) \;= \{add(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in TU_{\{s,0\},V}\}$

$\qquad\qquad = \{add(s^m(v), s^n(0), s^{n+m}(v)) \mid m, n \in \mathbb{N}, v \in V \cup \{0\}\}$

and the Herbrand interpretation

$\mathcal{M}(P_{add}) = \{add(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in HU_{\{s,0\}}\}$

$\qquad\qquad = \{add(s^m(0), s^n(0), s^{n+m}(0)) \mid m, n \in \mathbb{N}\}$

are models of P_{add}.

Correct Answer Substitutions versus Computed Answer Substitutions (Example)

Let P_{add} "add-program", and $Q = add(u, s(0), s(u))$ query.

- $\theta = \{u/s^2(v)\}$ correct answer substitution of Q, since $P_{add} \models Q\theta = add(s^2(v), s(0), s^3(v))$ (in analogy to slide 13 with $x = s^2(v)$).
- SLD-derivation of $P_{add} \cup \{Q\}$:

  $add(u, s(0), s(u)) \overset{\theta_1}{\Longrightarrow} add(u, 0, u) \overset{\theta_2}{\Longrightarrow} \square$ with $\theta_1 = \{x/u, y/0, z/u\}$ and $\theta_2 = \{x/u\}$,

  thus $\eta = (\theta_1\theta_2)|_{\{u\}} = \epsilon$ is a computed answer substitution of Q.
- Thus, $Q\eta$ more general than $Q\theta$.
- In fact, no SLD-derivation of $P_{add} \cup \{Q\}$ can deliver correct answer substitution θ.

Completeness of SLD-Resolution for Implication Trees

Query Q is n-deep.

$:\Leftrightarrow$

every atom in Q is the root of an implication tree,

and n is the total number of nodes in these trees

Lemma 4.15

Suppose $Q\theta$ is n-deep for some $n \geq 0$. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with CAS η such that $Q\eta$ is more general than $Q\theta$.

Completeness of SLD-Resolution (I)

Theorem 4.13

Suppose that θ is a correct answer substitution of Q. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with CAS η such that $Q\eta$ is more general than $Q\theta$.

Proof. Let $Q = A_1, ..., A_m$.Then: θ correct answer substitution of $A_1, ..., A_m$
implies $P \models A_1\theta, ..., A_m\theta$
implies for every interpretation I: if $I \models P$, then $I \models A_1\theta, ..., A_m\theta$
implies $\mathcal{C}(P) \models A_1\theta, ..., A_m\theta$ (since $\mathcal{C}(P) \models P$ by Lemma 4.12)
implies $inst(A_i\theta) \subseteq \mathcal{C}(P)$ for every $i = 1, ..., m$ (by Lemma 4.7)
implies $A_i\theta \in \mathcal{C}(P)$ for every $i = 1, ..., m$
implies $A_1\theta, ..., A_m\theta$ is n-deep for some $n \geq 0$ (by def. of $\mathcal{C}(P)$)
implies claim (by Lemma 4.15)

Completeness of SLD-Resolution (II)

Corollary 4.16

Suppose $P \models \exists Q$.

Then there exists a successful SLD-derivation of $P \cup \{Q\}$.

Proof. $P \models \exists Q$

implies $P \models Q\theta$ for some substitution θ

implies θ correct answer substitution of Q

implies claim (by Theorem 4.13)

Least Herbrand Model

Theorem 4.29 $\mathcal{M}(P)$ is the least Herbrand model of P.

Proof. Let I be a Herbrand model of P and let $A \in \mathcal{M}(P)$.

We proof $A \in I$ by induction on the number i of nodes in the ground implication tree w.r.t. P with root A. Then $\mathcal{M}(P) \subseteq I$.

$i = 1$: A leaf implies $A \leftarrow\; \in \mathit{ground}(P)$
implies $I \models A$ (since $I \models P$)
implies $A \in I$

$i \rightsquigarrow i+1$: A has direct descendants $B_1, \ldots, B_n$ (roots of subtrees)

implies $A \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ and $B_1, \ldots, B_n \in I$ (induction hypothesis)

implies $A \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ and $I \models B_1, \ldots, B_n$

implies $I \models A$ (since $I \models P$)

implies $A \in I$

Ground Equivalence

Theorem 4.30 For every ground atom A: $P \models A$ iff $\mathcal{M}(P) \models A$.

Proof. "⇒": $P \models A$ and $\mathcal{M}(P) \models P$ implies $\mathcal{M}(P) \models A$ (semantic consequence).

"⇐": Show for every interpretation I: $I \models P$ implies $I \models A$.
Let $I_H = \{A \mid A$ ground atom and $I \models A\}$ Herbrand interpretation.

$I \models P$

implies $I \models A \leftarrow B_1, ..., B_n$ for all $A \leftarrow B_1, ..., B_n \in ground(P)$

implies if $I \models B_1, ..., I \models B_n$ then $I \models A$ for all ...

implies if $B_1 \in I_H, ..., B_n \in I_H$ then $A \in I_H$ for all ... (Def. I_H)

implies $I_H \models P$ (by Lemma 4.26; thus I_H Herbrand model)

implies $A \in I_H$ (since $A \in \mathcal{M}(P)$ and $\mathcal{M}(P)$ least Herbrand model)

implies $I \models A$ (by Def. I_H)

Complete Partial Orderings

Let $(\mathcal{A}, \sqsubseteq)$ be a partial ordering (cf. Slide 18 for Chapter 2).

- a least element of $X \subseteq \mathcal{A}$

  $:\Leftrightarrow a \in X, a \sqsubseteq x$ for all $x \in X$

- a least upper bound of $X \subseteq \mathcal{A}$ (Notation: $a = \sqcup X$)

  $:\Leftrightarrow a \in \mathcal{A}, x \sqsubseteq a$ for all $x \in X$ and a is the least element of $\mathcal{A}$ with this property

$(\mathcal{A}, \sqsubseteq)$ complete partial ordering (CPO) $:\Leftrightarrow$

- $\mathcal{A}$ contains a least element (denoted by $\emptyset$)

- for every increasing sequence $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ of elements of $\mathcal{A}$,

  the set $X = \{a_0, a_1, a_2, \ldots\}$ has a least upper bound

Some Properties of Operators

Let $(\mathcal{A}, \sqsubseteq)$ be a CPO.

operator $T: \mathcal{A} \rightarrow \mathcal{A}$ monotonic

$:\Leftrightarrow$ $I \sqsubseteq J$ implies $T(I) \sqsubseteq T(J)$

operator $T: \mathcal{A} \rightarrow \mathcal{A}$ finitary

$:\Leftrightarrow$ for every infinite sequence $I_0 \sqsubseteq I_1 \sqsubseteq \ldots$,

$$\bigsqcup_{n=0}^{\infty} T(I_n) \textit{ exists} \qquad \textit{and} \qquad T\left(\bigsqcup_{n=0}^{\infty} I_n\right) \sqsubseteq \bigsqcup_{n=0}^{\infty} T(I_n)$$

operator $T: \mathcal{A} \rightarrow \mathcal{A}$ continuous $:\Leftrightarrow$ T monotonic and finitary

I pre-fixpoint of T $:\Leftrightarrow$ $T(I) \sqsubseteq I$

I fixpoint of T $:\Leftrightarrow$ $T(I) = I$

Iterating Operators

Let $(\mathcal{A}, \sqsubseteq)$ be a CPO, T: $\mathcal{A} \to \mathcal{A}$, and $I \in \mathcal{A}$.

- $T{\uparrow}0\ (I) :\Leftrightarrow I$
- $T{\uparrow}(n + 1)\ (I) :\Leftrightarrow T(T{\uparrow}n\ (I))$
- $T{\uparrow}\omega\ (I) :\Leftrightarrow \bigsqcup_{n=0}^{\infty} T{\uparrow}n\ (I)$

$T{\uparrow}\alpha :\Leftrightarrow T{\uparrow}\alpha\ (\emptyset)$ (for α = 0, 1, 2, ..., ω)

By the definition of a CPO:
If the sequence $T{\uparrow}0\ (I)$, $T{\uparrow}1\ (I)$, $T{\uparrow}2\ (I)$, ... is increasing, then $T{\uparrow}\omega\ (I)$ exists.

Theorem 4.22

If T is a continuous operator on a CPO, then $T{\uparrow}\omega$ exists and is the least pre-fixpoint of T and the least fixpoint of T.

Consequence Operator

Consider the CPO $(\{I \mid I \text{ Herbrand interpretation}\}, \subseteq)$.

Let P be a program and I a Herbrand interpretation. Then

$T_P(I) :\Leftrightarrow \{A \mid A \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P), \{B_1, \ldots, B_n\} \subseteq I\}$

Lemma 4.33

(i) T_P is finitary.

(ii) T_P is monotonic.

T_P-Characterization

Lemma 4.32

A Herbrand interpretation I is a model of P iff

$$T_P(I) \subseteq I$$

Proof.

$I \models P$

iff for every $A \leftarrow B_1, ..., B_n \in ground(P)$:

$\{B_1, ..., B_n\} \subseteq I$ implies $A \in I$ (by Lemma 4.26)

iff for every ground atom A: $A \in T_P(I)$ implies $A \in I$

iff $T_P(I) \subseteq I$

Characterization Theorem

Theorem 4.34

$$
\begin{array}{ll}
\mathcal{M}(P) & \text{(i)} \\
= \text{least Herbrand model of } P & \text{(ii)} \\
= \text{least pre-fixpoint of } T_P & \text{(iii)} \\
= \text{least fixpoint of } T_P & \text{(iv)} \\
= T_P{\uparrow}\omega & \text{(v)} \\
= \{A \mid A \text{ ground atom, } P \models A\} & \text{(vi)}
\end{array}
$$

Success Sets

success set of a program P :$\Leftrightarrow$
$\{A \mid A$ ground atom, $\exists$ successful SLD-derivation of $P \cup \{A\}\ \}$

Theorem 4.37

For a ground atom A, the following are equivalent:

(i) $\mathcal{M}(P) \models A$

(ii) $P \models A$

(iii) Every SLD-tree for $P \cup \{A\}$ is successful

(iv) A is in the success set of P

Objectives

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models