Coinductive logic programming

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Fixpoint Theorem

Fixed Point Theorem (Knaster, Tarski):
If $g : 2^S \to 2^S$ is monoton with respect to the subset ordering $\subseteq$ then g

1. has a least fixed point with respect to $\subseteq$ given by $\bigcap\{X \subseteq S \mid g(X) \subseteq X\}$
2. has a greatest fixed point with respect to $\subseteq$ given by $\bigcup\{X \subseteq S \mid X \subseteq g(X)\}$.

Proof: We will give only the proof of the first part, since the second part can be proved analogously.

Let denote

$$PRE_g = \{X \subseteq S \mid g(X) \subseteq X\}$$

respectively

$$POST_g = \{X \subseteq S \mid X \subseteq g(X)\}$$

the set of prefixed respectively postfixed points of the monoton mapping $g : 2^S \to 2^S$.

If we show that $\cap PRE_g \in PRE_g \cap POST_g$, then it is shown, that $\cap PRE_g$ is the least fixed point of g.Analogously, one has to prove that $\cup POST_g \in PRE_g \cap POST_g$ in order to prove that $\cup POST_g$ is the greatest fixed point of g.

First we check if PRE_g is empty. Because of $g(X) \subseteq S$ for each $X \subseteq S$ we obtain $g(S) \subseteq S$ and $S \in PRE_g$.

Because of $\cap PRE_g \subseteq X$ for each $X \in PRE_g$ it follows

$$g(\cap PRE_g) \subseteq g(X) \subseteq X \text{ for each } X \in PRE_g.$$

This implies $g(\cap PRE_g) \subseteq \cap PRE_g$, i.e. $\cap PRE_g \in PRE_g$.

If $X \in PRE_g$, then $g(X) \in PRE_g$, since $g(g(X)) \subseteq g(X)$ because of $g(X) \subseteq X$ and g monoton. This implies $g(\cap PRE_g) \in PRE_G$ since we proved $\cap PRE_g \in PRE_g$ above.

By definition of the intersection this finally implies $\cap PRE_g \subseteq g(\cap PRE_g)$, such that $\cap PRE_g \in POST_g$.

Now let be given a program P, Π the ranked alphabet of predicate symbols in P and F the ranked alphabet of function symbols in P..

Let $HU_F^\infty = TU_{F,\emptyset}^\infty$ be the infinitary Herbrand universe and $HB_{\Pi,F}^\infty = TB_{\Pi,F,\emptyset}^\infty$ the infinitary Herbrand base associated with P. $ground(P)$ denotes the set of all ground instances of clauses of P.

The consequence operator

$$T_P(X) = \{A \mid A \leftarrow B_1, \ldots, B_n \in ground(P), B_1, \ldots, B_n \in X\}$$

is a monoton mapping.

We know

- $I \subseteq HB_{\Pi,F} \subseteq HB^{\infty}_{\Pi,F}$ is a model of P if and only if $T_P(I) \subseteq I$, i.e. if I is a pre–fixed point of the consequence operator.
- The declarative semantics of P is given by the least fixed point $lfp(T_P)$ of the consequence operator.

The coinductive semantics of a program P can now be defined as follows:

- $I \subseteq HB^{\infty}_{\Pi,F}$ is a co–model if $I \subseteq T_P(I)$, i.e. if it is a post–fixed point of the consequence operator.
- The coinductive semantics of a program is given by the greatest fixed point $gfp(T_P)$ of the consequence operator.

We will illustrate the differences between the traditional and the coinductive semantics of a logical programm by a simple example:

```
stream([H|T]) :- number(H), stream(T).
number(0).
number(s(N)) :- number(N).

?- stream([0, s(0), s(s(0))|T]).
```

In the least fixed point semantics the question leads to `false`, since the least model assigns the empty set to the predicate `stream` and the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers to the predicate `number`.

What happens in the greatest fixed point semantics with the program

```
stream([H|T]) :- number(H), stream(T).
number(0).
number(s(N)) :- number(N).

?- stream([0, s(0), s(s(0))|T]).
```

Now the answer becomes `true` since

$$\begin{aligned}\{0,1,2,0,1,2,0,1,2,0,1,2,\ldots,\\ 1,2,0,1,2,0,1,2,0,1,2,\ldots,\\ 2,0,1,2,0,1,2,0,1,2,\ldots\}\end{aligned}$$

defines a co–model of the program above which assigns the set $\{0, 1, 2, 3, \ldots\} \cup \{\omega\}$ to the predicate `number` the three–element set above to the predicate `stream` and $gfp(T_P)$ is the union of all co–models.