1.5 Compactness Property and Löwenheim-Skolem Theorems

The fact that FOL has sound and complete deductive calculi has several important consequences. We start with the observation that finite satisfiability agrees with satisfiability. (Recall that \( \mathcal{F} \) is called finitely satisfiable if each finite subset of \( \mathcal{F} \) is satisfiable.) This property is known as compactness property.

**Theorem 1.5.1 (Compactness Theorem).** Let \( \mathcal{F} \) be a set of FOL-formulas. Then, \( \mathcal{F} \) is satisfiable iff every finite subset of \( \mathcal{F} \) is satisfiable.

**Proof.** “\( \Rightarrow \)”: obvious, as any model for \( \mathcal{F} \) is also a model for all its subsets.

“\( \Leftarrow \)”: Assume by contradiction that \( \mathcal{F} \) is not satisfiable, while any finite subset of \( \mathcal{F} \) has a model. Let \( \mathcal{D} \) be a sound and complete proof system for FOL. As \( \mathcal{F} \) is supposed to be unsatisfiable, we have:

\[
\mathcal{F} \vdash_{\mathcal{D}} \text{false}
\]

As \( \mathcal{D} \) is complete, we get:

\[
\mathcal{F} \vdash_{\mathcal{D}} \text{false},
\]

which means that there exists a \( \mathcal{D} \)-proof \( \psi_1, \ldots, \psi_m \) of \( \psi_m = \text{false} \) from \( \mathcal{F} \). In this \( \mathcal{D} \)-proof only finitely many formulas \( \psi_i \in \mathcal{F} \) appear. Hence, there is a finite subset \( \mathcal{F}' \) of \( \mathcal{F} \) with

\[
\mathcal{F}' \vdash_{\mathcal{D}} \text{false}.
\]

(Just consider \( \mathcal{F}' \overset{\text{def}}{=} \mathcal{F} \cap \{\psi_1, \ldots, \psi_m\} \).) But then \( \mathcal{F}' \vdash_{\mathcal{D}} \text{false} \) as \( \mathcal{D} \) is sound. Hence, \( \mathcal{F}' \) is not satisfiable. This contradicts the assumption.

**Corollary 1.5.2 (Compactness Theorem for the consequence relation).** Let \( \mathcal{F} \cup \{\phi\} \) be a set of FOL-formulas. Then: \( \mathcal{F} \vdash \phi \) if and only if \( \mathcal{G} \vdash \phi \) for some finite subset \( \mathcal{G} \) of \( \mathcal{F} \).

**Proof.** Using the compactness theorem (Theorem 1.5.1), the argument is as follows:

\[
\mathcal{F} \vdash \phi
\]

iff \( \mathcal{F} \cup \{\neg \phi\} \) is not satisfiable

iff \( \mathcal{F} \cup \{\neg \phi\} \) is not finitely satisfiable

iff there exists a finite subset \( \mathcal{G} \) of \( \mathcal{F} \) such that \( \mathcal{G} \cup \{\neg \phi\} \) is not satisfiable

iff there exists a finite subset \( \mathcal{G} \) of \( \mathcal{F} \) such that \( \mathcal{G} \vdash \phi \)

Here is a direct (alternative) argument for the implication “\( \Rightarrow \)” in Corollary 1.5.2. If \( \mathcal{F} \vdash \phi \) then

\[
\mathcal{F} \vdash_{\mathcal{D}} \phi
\]

where \( \mathcal{D} \) is a sound and complete deductive calculus for FOL. Since any \( \mathcal{D} \)-proofs from \( \mathcal{F} \) is finite and only uses finitely many formulas in \( \mathcal{F} \) we get that

\[
\mathcal{G} \vdash_{\mathcal{D}} \phi
\]

for some finite subset \( \mathcal{G} \) of \( \mathcal{F} \). But then \( \mathcal{G} \vdash \phi \) (as \( \mathcal{D} \) is sound). 

67
Another characteristic feature of FOL are the Löwenheim-Skolem Theorems that state the existence of models of a certain cardinality for satisfiable sets of formulas. Recall that the given vocabulary and set of variables are supposed to be recursively enumerable. This ensures that the set of all FOL-formulas and any subset thereof is countable.

**Theorem 1.5.3 (Upward Löwenheim-Skolem Theorem (from finite to infinite models)).** Let \( \mathcal{F} \) be a set of formulas such that for each \( n \in \mathbb{N} \) there exists a finite model \( A_n \) for \( \mathcal{F} \) with at least \( n \) elements. Then, \( \mathcal{F} \) has an infinite model.

**Proof.** For \( n \geq 2 \) let

\[
\psi_n \overset{\text{def}}{=} \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j)
\]

Then, \( \psi_n \) is a sentence which holds for exactly those structures \( A \) where the domain \( \text{Dom}^A = A \) contains at least \( n \) elements. We now consider the formula-sets

\[
\mathcal{F}_m \overset{\text{def}}{=} \mathcal{F} \cup \{ \psi_n : 2 \leq n \leq m \}
\]

for \( m \geq 2 \) and

\[
\mathcal{F}' \overset{\text{def}}{=} \bigcup_{m \geq 2} \mathcal{F}_m = \mathcal{F} \cup \{ \psi_n : n \geq 2 \}.
\]

Since \( \mathcal{F} \) is satisfiable over finite structures of arbitrary size, the formula-sets \( \mathcal{F}_m \) are satisfiable for all \( m \geq 2 \). Since each finite subset of \( \mathcal{F}' \) is contained in some \( \mathcal{F}_m \), the compactness theorem yields that \( \mathcal{F}' \) is satisfiable. Let \( A \) be a model for \( \mathcal{F}' \) and \( A = \text{Dom}^A \) its domain. Since \( \mathcal{F} \subseteq \mathcal{F}' \), \( A \) is a model for \( \mathcal{F} \) too. But \( A \models \{ \psi_n : n \in \mathbb{N}, n \geq 2 \} \) yields that \( A \) is infinite.

Note that this argument also works for the case where \( \mathcal{F} \) is a set of FOL-formulas without equality. Although the formulas \( \psi_n \) use the equality symbol, they just serve to construct an infinite model for \( \mathcal{F} \).

In the above theorem it is important that we require the existence of finite models of arbitrary size. From the existence of a finite model we cannot derive the existence of an infinite model. For example, the FOL-sentence

\[
\phi = \forall x \forall y. (x = y)
\]

is satisfiable and any structure with a singleton domain is a model for \( \phi \). But there is no infinite model for \( \phi \), even no model with two or more elements. However, for FOL without equality, satisfiable formula-sets have arbitrary large models. More precisely, we show in the next theorem that for FOL without equality the assumption that there are finite models of arbitrary size is irrelevant and infinite models can be guaranteed for all satisfiable sets of FOL-formulas without equality.

**Theorem 1.5.4 (Upward Löwenheim-Skolem Theorem for FOL without equality).** Let \( \mathcal{F} \) be a satisfiable set of FOL-formulas without equality and let \( C \) be a set. Then, \( \mathcal{F} \) is satisfiable over some structure where the domain is a superset of \( C \).

**Proof.** Let \( (B, \mathcal{W}) \) be a model for \( \mathcal{F} \) and let \( B \) be the domain of \( B \). W.l.o.g., \( B \cap C = \emptyset \). We define a structure \( A \) with domain

\[
A \overset{\text{def}}{=} B \cup C
\]

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as follows. To provide the interpretations of the predicate and function symbols, we fix an element $b_0 \in B$ and define a surjective function $h : A \to B$ by:

$$h(a) \overset{\text{def}}{=} \begin{cases} a & \text{if } a \in B \\ b_0 & \text{if } a \in C \end{cases}$$

For each $n$-ary predicate symbol $P$ we put

$$P^A \overset{\text{def}}{=} \{ (a_1, \ldots, a_n) \in B^n : (h(a_1), \ldots, h(a_n)) \in P^B \}.$$  

The function $f^A : A^m \to A$ for each $m$-ary function symbol $f$ is defined by:

$$f^A(a_1, \ldots, a_m) \overset{\text{def}}{=} f^B(h(a_1), \ldots, h(a_m))$$

Then, $f^A(\vec{a}) \in B$ and therefore:

$$h(f^A(a_1, \ldots, a_m)) = f^A(a_1, \ldots, a_m) = f^B(h(a_1), \ldots, h(a_m))$$

Furthermore, we define a variable valuation $V : \text{Var} \to A$ by $V(x) = W(x)$ for all variables $x \in \text{Var}$. (Note that $W(x) \in B \subseteq A$.) Hence, $h : A \to B$ is a surjective homomorphism from $A$ to $B$ and the given variable valuation $W : \text{Var} \to B$ has the form $W = h \circ V$. Since we deal here with FOL without equality, we have:

$$(A, V) \models \phi \text{ iff } (B, W) \models \phi$$

for all formulas $\phi$ (see page 11). Since $(B, W)$ is a model for $\mathcal{F}$, so is $(A, V)$. \hfill \Box

In particular, any satisfiable set of FOL-formulas without equality has infinite models. Note that this does not contradict the lack of the finite model property (Theorem 1.3.2 on page 31), which states that there are satisfiable formulas that do not have any finite model and implies that there are FOL-formulas that hold for all finite structures without being valid. The above theorem just asserts the existence of infinite models for satisfiable formulas.

As the formula $\forall x \forall y. (x = y)$ shows (see above), the statement of Theorem 1.5.4 does not hold for FOL with equality. However, the statement of the upward Löwenheim-Skolem Theorem also holds for FOL with equality and satisfiable sets with at least one infinite model.

**Theorem 1.5.5 (Upward Löwenheim-Skolem Theorem (from infinite to larger models)).**

Let $\mathcal{F}$ be a formula-set that is satisfiable over some infinite structure and let $C$ be a set. Then, $\mathcal{F}$ is satisfiable over some structure where the domain is a superset of $C$.

**Proof.** In this proof, we drop the requirement that vocabularies have to be recursively enumerable. This requirement is only needed when speaking about algorithmic aspects, but it is irrelevant for the existence of sound and complete proof systems for FOL and the compactness property.\footnote{The notions of axioms and proof rules a Hilbert proof system have to be refined to treat formulas over arbitrary (possibly not recursively enumerable or even uncountable) vocabularies, since the requirement that they stand for decidable sets of formulas or formula tuples, respectively, is not adequate anymore. However, the axioms and proof rules are typically independent from some concrete vocabularies and rely on schemata that only use the abstract} Let $\text{Voc}$ be the underlying vocabulary of formula-set $\mathcal{F}$. We now regard the vocabulary $\text{Voc}_C$ which extends $\text{Voc}$ by fresh constant symbols $c \in C$, where we assume w.l.o.g. that none of the elements in $C$ appears in $\text{Voc}$. Let

$$\mathcal{G} \overset{\text{def}}{=} \mathcal{F} \cup \{ \text{“}c \neq d\text{”} : c, d \in C, c \neq d \}$$

69
where “c ≠ d” is viewed as formula over VocC.

Claim. ∅ is satisfiable.

Proof of the claim. For this, we take a finite subset ∅₀ of ∅ and show that ∅₀ is satisfiable. There exists pairwise distinct elements c₁,…,cₙ ∈ C such that

∅₀ ⊆ ∅ ∪ (“cᵢ ≠ cⱼ”: 1 ≤ i,j ≤ n, i ≠ j).

where “cᵢ ≠ cⱼ” is viewed as a literal over VocC. Let (B,W) be a model for ∅ with an infinite domain B. We pick pairwise distinct elements b₁,…,bₙ ∈ B and define a new structure B₀ which extends B. The domain of structure B₀ is B, which permits to define B₀’s meaning of the predicate and function symbols as in A, i.e.,

pᴮ₀ ≡ pᴮ and fᴮ₀ ≡ fᴮ

for each predicate symbol P and function symbol f in Voc. For the constant symbols c₁,…,cₙ we define:

cᵢᴮ₀ := bᵢ for 1 ≤ i ≤ n.

Obviously, (B₀,W) is a model for ∅ (as (B,W) is) and a model for the formulas “cᵢ ≠ cⱼ” for 1 ≤ i,j ≤ n, i ≠ j. Hence:

(B₀,W) |= ∅₀

This completes the proof of the claim.

We now take a model (A,V) for ∅. Let A be the domain of A. Since A is a model for the formulas “c ≠ d” where c, d ∈ C, c ≠ d, the elements cⱼ ∈ A for c ∈ C are pairwise distinct. Thus, we may assume that C ⊆ A.

Hence, any set ∅ of FOL-formulas that has some infinite model A then for each cardinal κ larger than the cardinality of (the domain of) A, there is a model B for ∅ of cardinality at least κ. To see this, we take C = 2ᴬ in Theorem 1.5.5 where A is the domain of A. For instance, any formula set that has an infinite countable model also has a model where the domain is a superset of ℝ or a superset of the powerset of the reals or a superset of the powerset of the powerset of the reals, and so on. We state here without proof that, given an infinite model A for ∅, then for each cardinal κ larger than the cardinality of A there is a model B for ∅ of cardinality (exactly) κ. This yields, e.g., that any formula-set that has a model where the domain is N also has a model with domain ℝ and a model with domain 2ᴺ, and so on.

The following theorem shows that each satisfiable set of FOL-formulas has a countable model. For FOL without equality, we can even guarantee infinite countable models. To establish this

formula symbols as atoms, but no atomic formulas P(t₁,…,tₙ) with a concrete predicate symbol P. Thus, we may redefine axioms and proof rules as decidable sets of (tuples of) formulas build by abstract formula symbols (rather than atomic formulas over some concrete vocabulary), the constant true, the boolean connectives ¬, ∧ and universal quantification ∀. Each such formula Φ with abstract formula symbols, say Ξ₁,…,Ξₙ stands for the set of all formulas Φ = Φ[Ξ₁/ψ₁,…,Ξₙ/ψₙ] over some concrete vocabulary Voc that arise by uniformly replacing the abstract formula symbols Ξᵢ that appear in Φ with formulas ψᵢ over Voc. For FOL with equality we also might need abstract term symbols T,S,… and abstract function symbols that range over all terms and function symbols, respectively. Then, e.g., one can specify the axiom consisting of all formulas t = s → s = t over some concrete vocabulary by the single formula T = S → S = T.
result it is important that we require the variable-set Var and the vocabulary Voc to be countable. (Our default-assumption that the underlying vocabulary Voc and the variable set Var are recursively enumerable is irrelevant. We just need here that Var and Voc are countable.)

**Theorem 1.5.6 (Downward Löwenheim-Skolem Theorem).** Each satisfiable set of FOL-formulas (over some countable vocabulary and variable-set) has a countable model. More precisely:

- Each satisfiable set of FOL-formulas without equality has an infinite countable model.
- Each satisfiable set of FOL-formulas with equality has a (finite or infinite) countable model.

**Proof.** For the special case of a singleton formula-set consisting of a closed FOL-formula in Skolem form, i.e., sentences of the form $\phi = \forall x_1 \ldots \forall x_n. \psi$ where $\psi$ is quantifier-free (see page 1.1), we may apply the Herbrand-theory, which yields that $\phi$ is satisfiable if and only if $\phi$ has a Herbrand-model. Since the Herbrand-universe is countable this yields the claim. For the general case, an analogous argument is applicable. The rough idea is to construct a term model, i.e., a model where the terms serve as elements of the domain. Note that the set of terms is countable since the vocabulary (i.e., set of predicate and function symbols) and the variable-set are supposed to be countable.

The idea is to extend the given formula-set $\mathcal{F}$ to a maximal satisfiable formula-set $\mathcal{F}^+$. More precisely, we construct a formula-set $\mathcal{F}^+$ which enjoys the following properties:

1. $\mathcal{F} \subseteq \mathcal{F}^+$ and $\mathcal{F}^+$ is satisfiable.
2. For each formula $\phi$: either $\phi \in \mathcal{F}^+$ or $\neg \phi \in \mathcal{F}^+$.
3. For each formula $\phi$ and variable $x$ there is a constant symbol $c$ such that $\neg \forall x. \phi \rightarrow \neg \phi[x/c] \in \mathcal{F}^+$.

**Construction of a term model for $\mathcal{F}^+$ for FOL without equality.** If this formula-set $\mathcal{F}^+$ has been constructed then we may define a structure $A$ and variable valuation $V$ as follows. We first explain the case where $\mathcal{F}$ is a set of FOL-formulas without equality. To ensure that the set of terms is infinite we suppose that the vocabulary has at least one function symbol of arity $\geq 1$. If this is not the case then we simply extend the given vocabulary by an unary function symbol. The domain $A$ of $A$ is the set of terms that can be built by the variables and constant and function symbols that appear in $\mathcal{F}^+$. Then, $A$ is countable (as Voc and Var are countable). Terms are interpreted by itself. I.e.:

- $V(x) \overset{\text{def}}{=} x$ for each variable $x$,
- $c^A \overset{\text{def}}{=} c$ for each constant symbol $c$,
- the function $f^A$ for an $m$-ary function symbol $f$ is given by $f^A(t_1, \ldots, t_m) \overset{\text{def}}{=} f(t_1, \ldots, t_m)$. 

71
The predicates $P^A$ for the $n$-ary predicate symbols are given by:

$$P^A \overset{\text{def}}{=} \{ (t_1,\ldots,t_n) \in A^n : P(t_1,\ldots,t_n) \in \mathcal{F}^+ \}$$

For the interpretation $\mathcal{I} = (A,V)$ we have $t^\mathcal{I} = t$ for all terms $t$. We now show:

$$\phi \in \mathcal{F}^+ \iff (A,V) \models \phi \quad (\star)$$

Obviously, $(\star)$ yields that $(A,V)$ is a model for $\mathcal{F}^+$, and hence, also for $\mathcal{F}$ (see condition (1)).

Proof of $(\star)$. We first observe that the maximal satisfiability of $\mathcal{F}^+$ (conditions (1) and (2)) yields that $\mathcal{F}^+$ is closed under consequences, i.e.:

$$\text{if } \mathcal{F}^+ \models \psi \text{ then } \psi \in \mathcal{F}^+.$$  

Note that if $\mathcal{J}$ is an interpretation that yields a model for $\mathcal{F}^+$ and $\mathcal{F}^+ \models \psi$ then $\mathcal{J} \models \psi$, and therefore, $\mathcal{J} \not\models \neg \psi$. But then $\neg \psi \not\in \mathcal{F}^+$. The maximality condition (2) yields $\psi \in \mathcal{F}^+$. In particular, $\mathcal{F}^+$ is closed under equivalence $\equiv$, i.e.:

$$\text{if } \phi \in \mathcal{F}^+ \text{ and } \phi \equiv \psi \text{ then } \psi \in \mathcal{F}^+.$$  

Furthermore, for all formulas $\psi, \psi_1, \psi_2$ we have:

$$\neg \psi \in \mathcal{F}^+ \iff \psi \not\in \mathcal{F}^+ \quad \text{ and } \quad \psi_1 \land \psi_2 \in \mathcal{F}^+ \iff \psi_1 \in \mathcal{F}^+ \text{ and } \psi_2 \in \mathcal{F}^+.$$  

The first statement is immediate from the maximal satisfiability of $\mathcal{F}^+$. The second statement follows by the facts that $\mathcal{F}^+$ is closed under consequences and $\psi_1 \land \psi_2 \models \psi_i$, $i = 1, 2$, and $\{\psi_1, \psi_2\} \models \psi_1 \land \psi_2$.

We now prove $(\star)$ by induction on the length $k$ of formulas.

Basis of induction ($k = |\phi| = 0$). For $\phi = \text{true}$ we have $\text{true} \in \mathcal{F}^+$ (since $\mathcal{F}^+$ is maximal satisfiable) and $(A,V) \models \text{true}$. We now regard an atomic formula $\phi = P(t_1,\ldots,t_n)$. then:

$$P(t_1,\ldots,t_n) \in \mathcal{F}^+ \iff (t_1,\ldots,t_n) \in P^A \quad \text{(by definition of } P^A) \quad \text{ and } \quad (A,V) \models P(t_1,\ldots,t_n).$$  

(As we deal with FOL without equality, we do not consider atomic formulas of the form $t_1 = t_2$.)

Induction step. Let $k = |\phi| \geq 1$ and let us suppose that $(\star)$ holds for all formulas of length $\leq k - 1$ (induction hypothesis). If $\phi = \neg \psi$ then

$$\neg \psi \in \mathcal{F}^+ \iff \psi \not\in \mathcal{F}^+ \quad \text{(as } \mathcal{F}^+ \text{ is maximal satisfiable)}$$

$$\iff (A,V) \not\models \psi \quad \text{(induction hypothesis)}$$

$$\iff (A,V) \models \neg \psi = \phi$$

72
The treatment of $\phi = \psi_1 \land \psi_2 \in \mathfrak{F}^+$ is analogous:

$$\psi_1 \land \psi_2 \in \mathfrak{F}^+$$

$$\iff \{\psi_1, \psi_2\} \subseteq \mathfrak{F}^+$$  \hspace{1cm} (as $\mathfrak{F}^+$ is maximal satisfiable)

$$\iff (A, V) \models \psi_1 \text{ and } (A, V) \models \psi_2$$  \hspace{1cm} (induction hypothesis)

$$\iff (A, V) \models \psi_1 \land \psi_2 = \phi$$

Let now $\phi = \forall y. \psi$.

"$\iff$": Suppose $\models (A, V) \models \forall y. \psi$. Then, $(A, V[y := s]) \models \psi$ for all terms $s$. In particular, we have:

$$(A, V[y := c]) \models \psi$$

where $c$ is a constant symbol as in condition (3) such that $\neg \forall y. \psi \rightarrow \neg \psi[y/c] \in \mathfrak{F}^+$. The substitution lemma (see page 9) yields:

$$(A, V) \models \psi[y/c]$$

Since $|\psi[y/c]| = |\psi| - 1 = k - 1$ we may apply the induction hypothesis and get:

$$\psi[y/c] \in \mathfrak{F}^+$$

By the choice of $c$ we have $\neg \forall y. \psi \rightarrow \neg \psi[y/c] \in \mathfrak{F}^+$. As

$$\neg \forall y. \psi \rightarrow \neg \psi[y/c] \equiv \neg \forall y. \psi \lor \neg \psi[y/c]$$

$$\equiv \psi[y/c] \lor \forall y. \psi$$

we have:

$$\{\psi[y/c], \neg \forall y. \psi \rightarrow \neg \psi[y/c]\} \models \forall y. \psi$$

Since $\mathfrak{F}^+$ is closed under consequences (see above) we get $\forall y. \psi \in \mathfrak{F}^+$.

"$\Rightarrow$": Suppose now that $\forall y. \psi \in \mathfrak{F}^+$. We have to show that $\models (A, V[y := s]) \models \psi$ for all terms $s \in A$. In the sequel, let $s$ be a fixed term. By renaming bounded variables of $\psi$, we switch from $\psi$ to an equivalent formula $\psi'$ of the same length such that no variable that appears in $s$ has some bounded occurrences in $\psi'$. This ensures that $y$ can be replaced with $s$ in $\psi'$.

As $\forall y. \psi \equiv \forall y. \psi'$ and as we suppose that $\forall y. \psi \in \mathfrak{F}^+$ we also have $\forall y. \psi' \in \mathfrak{F}^+$. Since variable $y$ can be replaced in $\psi'$ with the term $s$ we have:

$$\forall y. \psi' \models \psi'[y/s]$$

and therefore:

$$\psi'[y/s] \in \mathfrak{F}^+$$

73
Since the length of $\psi'[y/s]$ agrees with the length of $\psi'$ and $\psi$ and is smaller than the length of $\phi = \forall y.\psi$ (more precisely, we have $|\psi'[y/s]| = |\psi'| = |\psi| = k - 1$), we may apply the induction hypothesis which yields:

$$\langle A, \mathcal{V} \rangle \models \psi'[y/s]$$

As variable $y$ can be replaced with term $s$ in $\psi'$ we get by the substitution lemma (see page 9):

$$\langle A, \mathcal{V} \rangle \models \psi'$$

This shows $\langle A, \mathcal{V}[y := s]\rangle \models \psi$ for all terms $s$, and therefore $\langle A, \mathcal{V} \rangle \models \forall y.\psi$.

**Construction of a term model for $\mathfrak{F}^+$ for FOL with equality.** To treat sets of FOL-formulas with equality, a countable (possibly finite) model for $\mathfrak{F}^+$ is obtained by the quotient structure $A/\sim$ where $A$ is above and $\sim$ denotes the equivalence that identifies exactly those terms $t_1$ and $t_2$ such that $\mathfrak{F}^+ \vdash (t_1 = t_2)$. We then have:

$$t_1 \sim t_2 \iff \mathfrak{F}^+ \vdash (t_1 = t_2) \iff t_1 = t_2 \in \mathfrak{F}^+$$

The quotient structure $A/\sim$ is obtained as follows. The domain of $A/\sim$ is the quotient space of $A$ with respect to $\sim$, i.e., $\text{Dom}^{A/\sim}$ equals

$$A/\sim = \{ [t] : t \in A \}$$

where

$$[t] \defeq \{ t' : t \sim t' \}$$

denotes the equivalence class of $t$. For each $n$-ary predicate symbol $P$ we put

$$P^{A/\sim} \defeq \{ ([t_1], \ldots, [t_n]) : P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \}.$$

Note that, since $\mathfrak{F}^+$ is maximal satisfiable, we have for all terms $t_1, \ldots, t_n, s_1, \ldots, s_n$:

if $t_i \sim s_i$ for $i = 1, \ldots, n$ then:

$$P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \iff P(s_1, \ldots, s_n) \in \mathfrak{F}^+$$

Thus, if $T_1, \ldots, T_n \in A/\sim$ then

$$(T_1, \ldots, T_n) \in P^{A/\sim} \iff P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \text{ for all terms } t_i \in T_i, i = 1, \ldots, n.$$}

Similarly, if $f$ is an $m$-ary function symbol then we define:

$$f^{A/\sim}([t_1], \ldots, [t_m]) \defeq [f(t_1, \ldots, t_m)]$$

This function $f^{A/\sim}$ is well-defined since for all terms $t_1, \ldots, t_m, s_1, \ldots, s_m$ such that $t_i \sim s_i$ for $1 \leq i \leq m$ we have $f(t_1, \ldots, t_m) \sim f(s_1, \ldots, s_m)$. The variable valuation $\mathcal{V}' : \text{Var} \to A/\sim$ is given by $\mathcal{V}'(x) \defeq [x]$. It is now easy to see that

$$\langle A/\sim, \mathcal{V}' \rangle \models \phi \iff \phi \in \mathfrak{F}^+$$

In particular, $\langle A/\sim, \mathcal{V}' \rangle$ is a countable (possibly finite) model for $\mathfrak{F}^+$, and hence also for $\mathfrak{F}$. 74