Trace Machines for Observing Continuous-Time Markov Chains

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Abstract

In this paper, we study several linear-time equivalences (Markovian trace equivalence, failure and ready trace equivalence) for continuous-time Markov chains that refer to the probabilities for timed execution paths. Our focus is on testing scenarios by means of push-button experiments with appropriate *trace machines* and a discussion of the connections between the equivalences. For Markovian trace equivalence, we provide alternative characterizations, including one that abstracts away from the time instances where actions are observed, but just reports on the average sojourn times in the states. This result is used for a reduction of the question whether two finite-state continuous-time Markov chains are Markovian trace equivalent to the probabilistic trace equivalence problem for discrete-time Markov chains (and the latter is known to be solvable in polynomial time).

1 Introduction

In the past 15 years, various process calculi and temporal logics have been developed for the design and analysis of stochastic systems (see e.g. [3] for an overview) and applied e.g. for reasoning about communication systems, biological systems or mobile multi-agent systems, see e.g. [20,14,12,7,21].

The model used in this paper are continuous-time Markov chains with action labels (ACTMC for short). ACTMCs are widely used as operational model for stochastic process algebras, such as [2,15,9,22]. They can be viewed as ordinary labelled transition systems where the states are augmented with *exit rates* that specify the (average) residence times in the states.

The main contribution of our paper are testing scenarios for Markovian trace equivalence on ACTMCs. Intuitively, two ACTMCs are considered "equal" if they can not be distinguished by observing the sequence of performed actions (*trace*) in infinitely many runs. Assume, that an ACTMC M is modelled as a black box (*Markovian trace machine*, illustrated in Fig. 1.) with two information displays and one button such that

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- the first display shows the action currently performed by M,
- the second display shows some time information, for example the absolute time,
- the reset button restarts the process, i.e. the ACTMC M starts another run, if the reset button is pressed.

The information on the displays (*observation*) is recorded by an external observer and at an arbitrary time instant she decides to push the reset button to observe another run. After infinitely many runs the probability of each possible observation can be calculated. Let $O_{\mathcal{A}}$ and $O_{\mathcal{T}}$ be the set of all possible observations on the action display and on the time display, respectively. For two ACTMCs M, M' and observation $o \in O_{\mathcal{A}} \times O_{\mathcal{T}}$ we check if the probability that o occurs while testing M equals the probability that o is observed while testing M'. M and M' are *equivalent* if the probabilities coincide for all possible observations.
We consider different variants of the Markovian trace machine.

(i) The time display shows the absolute time.

(ii) For each transition the observer provides a value for a hidden countdown timer that starts at the time instant when an action is executed. The time display shows if this timer expired before the next action is performed or not. Hence, one knows if the time duration between the two successive actions is lower or equal to the value of the countdown timer.

(iii) The action display shows the $\perp-$symbol if the process reaches a deadlock state, i.e. the observer can distinguish between a run that ends up in a deadlock state and a run that is finished by herself because of a restart.

(iv) The trace machine is equipped with *action lights* that give information about the set of actions that are possible at a certain time instant during a run, i.e. that have a non-zero probability.

Beside the trace-machine-approach we provide an alternative characterization for Markovian trace equivalence that abstracts away from the concrete observable timing behaviours and just refers to the average sojourn times of the

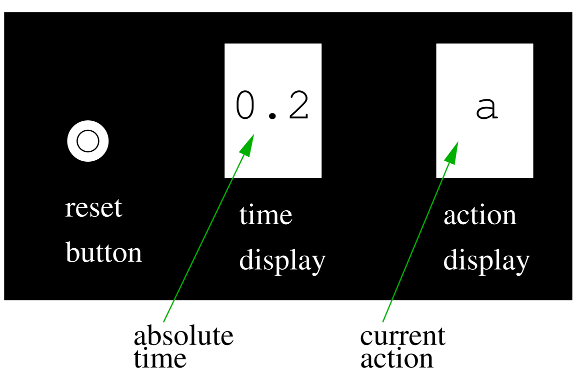

Fig. 1. Markovian Trace Machine.

intermediate states (in form of their exit rates). This characterization is easier and thus helpful for formal reasoning with Markovian trace equivalence. In addition, for finite-state ACTMCs it allows to reduce the problem of whether two ACTMCs are Markovian trace equivalent to the discrete-time case, where the corresponding notion of probabilistic trace equivalence is known to be decidable in polynomial time [16]. Finally, we discuss Markovian variants of failure and ready trace equivalence which turn out to agree and to be refinements of Markovian trace equivalence.

Related work. Although various process equivalences and preorders have been defined for discrete-time or time-abstract probabilistic models, such as (bi)simulation relations [19,28,10,25], testing relations [6,5,17,24,26] probabilistic trace equivalences [16,23], in the continuous-time setting research has mainly concentrated on branching time relations [15,2,13,4]. Publications on linear time relations and testing scenarios for continuous-time Markov models are rare. An exception is the work by Bernardo and Cleaveland [1] where testing relations are studied for EMPA-processes. Unlike [1], where testing relations are defined by "test processes" that interact with the system under consideration, we aim at testing scenarios for ACTMCs by means of push-button experiments with a machine model in the style of [27] and [26] for time-abstract probabilistic automata. As pointed out in [26] the use of machine models yields a *fully observable* characterization of the system which does not only describe the observable execution paths but also how the probabilities for a successful test can be observed.

Organization of the paper. Section 2 contains some preliminaries concerning exponential distributions. In Section 3, we introduce action-labelled continuous-time Markov chains. Markovian trace equivalence is studied in Section 4, while failure and ready trace equivalence are considered in Section 5. The paper ends with a brief summary in Section 6.

Throughout the paper, we assume some familiarity with basic notions of probability theory and Markov chains, see e.g. [18,11].

2 Preliminaries: exponential distributions

Let X be a continuous random variable and $q \in \mathbb{R}_{>0}$. X is exponentially distributed with parameter q if the probability that X is less or equal $t \in \mathbb{R}_{\geq 0}$ is given by $Expo(q,t) = 1 - e^{-q \cdot t}$. We put $expo(q,t) = \frac{d}{dt} Expo(q,t) = q \cdot e^{-q \cdot t}$. For a function $g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we write $g(x,\cdot)$ to denote the function $y \mapsto g(x,y)$. The convolution $f * h$ of two continuous functions $f,h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

$$[f * h](t) = \int_0^\infty f(x) \cdot h(t-x) dx.$$

Let $(X_1, X_2, \ldots, X_{n+1})$ be a vector of independent random variables such that X_i is exponentially distributed with parameter $q_i > 0$, $1 \leq i \leq n+1$. Then for $\alpha = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n_{\geq 0}$ and $\gamma = (q_1, q_2, \ldots, q_n)$ we define

$$Expo(\gamma, \alpha) = Prob\,\{X_1 \leq t_1, X_2 \leq t_2, \ldots, X_n \leq t_n\} = \prod_{i=1}^{n} Expo(q_i, t_i).$$

For $\gamma' = (q_1, q_2, \ldots, q_{n+1})$, $\alpha' = (t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1}_{\geq 0}$, $y_j = \sum_{i=1}^{j} t_i$ let

$$\begin{aligned} Conv(\gamma', \alpha') &= Prob\left\{\sum_{i=1}^{j-1} X_i \leq y_j < \sum_{i=1}^{j} X_i, 1 \leq j \leq n+1\right\} \\ &= (1 - Expo(q_1, t_1)) \cdot \prod_{j=1}^{n} \left[expo(q_j, \cdot) * (1 - Expo(q_{j+1}, \cdot))\right](t_{j+1}). \end{aligned}$$

3 Action-labelled continuous-time Markov chains

Let *Act* be a non-empty finite set of actions.

Definition 3.1 [**ACTMC**] An *action-labelled continuous-time Markov chain* M is a triple $(S, \rightarrow, s_{init})$ where S is a countable set of states, $s_{init} \in S$ is the initial state and $\rightarrow \subseteq S \times (Act \times \mathbb{R}_{>0}) \times S$ is a transition relation such that for all $s \in S$ the set of transitions $(s, a, \lambda, s') \in\rightarrow$ is countable and the *exit rate* is finite, i.e.

$$q(s) = \sum_{s', a, \lambda : (s, a, \lambda, s') \in \longrightarrow} \lambda \; < \infty.$$

Here and in the sequel, we simply write $s \xrightarrow{(a,\lambda)} s'$ instead of $(s, a, \lambda, s') \in\rightarrow$.
□

The parameter λ of a transition $s \xrightarrow{(a,\lambda)} s'$ specifies the delay of this transition. As soon as state s is entered the probability for $s \xrightarrow{(a,\lambda)} s'$ to be enabled after at most t time units is $1 - e^{-\lambda t}$. If there are more than one outgoing transitions from state s then the transition which is first enabled is taken. Thus, the time an ACTMC M spends in state s is exponentially distributed with parameter $q(s)$, i.e. $Expo(q(s), t) = 1 - e^{-q(s)t}$ is the probability that M remains at most t time units in s. If $q(s) = 0$ the process remains infinitely long in s, in which case state s can be regarded as a deadlock state and $expo(0, t) = Expo(0, t) = 0$ for all $t \geq 0$.

For a given ACTMC $M = (S, \rightarrow, s_{init})$, $s, s' \in S$, $S' \subseteq S$ and $a \in Act$,

$$r(s, a, s') = \sum_{s \xrightarrow{(a,\lambda)} s'} \lambda \quad \text{and} \quad p(s, a, S') = \begin{cases} \sum_{s' \in S'} r(s, a, s')/q(s) & \text{if } q(s) > 0, \\ 0 & \text{if } q(s) = 0. \end{cases}$$

We often omit brackets, e.g. we write $p(s, a, s')$ for $p(s, a, \{s'\})$ and if M is not clear from the context we add a superscript M, e.g. $r^M(s, a, s') = r(s, a, s')$. Note that $r(s, a, s')$ is the total rate to move from s via an a-transition to state

s'. Thus, $p(s, a, s')$ denotes the (time-abstract) probability to move from state s via an a-transition to state s'.

Definition 3.2 [Paths in ACTMCs] Let $M = (S, \rightarrow, s_{init})$ be an ACTMC. A *path* π in M is a finite sequence

$$\pi = \langle s_0, a_1, s_1, a_2 \ldots s_{n-1}, a_n, s_n \rangle \in (S \times Act)^* \times S \text{ with } r(s_i, a_{i+1}, s_{i+1}) > 0.$$

Let $|\pi| = n$ be the length of π, $trace(\pi) = a_1, a_2, \ldots, a_n$ and $last(\pi) = s_n$ and $first(\pi) = s_0$ the last and the first state on π, respectively. We write $Path(s)$ for the set of all paths π with $first(\pi) = s$ and $Path(s, \sigma)$ for the set of all paths $\pi \in Path(s)$ with $trace(\pi) = \sigma$. □

Note that if $\sigma = \langle a_1, \ldots, a_n \rangle$ then $Path(s, \sigma)$ contains only paths of length n. The (discrete-time) probability $dp(\pi)$ of path π is given by

$$dp(\pi) = \prod_{i=0}^{n} p(s_i, a_{i+1}, s_{i+1}).$$

It is easy to verify that for all $s \in S$, $n \in \mathbb{N}$ we have $\sum_{\pi \in Path(s), |\pi| = n} dp(\pi) = 1$. Furthermore, we define

$$E(\pi) = (q(s_0), q(s_1), \ldots, q(s_{n-1})) \in \mathbb{R}^n_{\geq 0}$$

as the vector of exit rates of all states visited on π except $last(\pi)$ and

$$E'(\pi) = (q(s_0), q(s_1), \ldots, q(s_n)) \in \mathbb{R}^{n+1}_{\geq 0}$$

as the vector with all exit rates. For $E \in \mathbb{R}^n_{\geq 0}$ and $E' \in \mathbb{R}^{n+1}_{\geq 0}$, let $Path(s, \sigma, E)$ and $Path(s, \sigma, E')$ denote the set of all $\pi \in Path(s, \sigma)$ with $E(\pi) = E$ and $E'(\pi) = E'$, respectively.

4 Markovian Trace Equivalence

In the following, we discuss several notions of trace equivalences on ACTMCs based on different variants of the Markovian trace machine (cf. Fig. 1). In either case, the probability to observe an action sequence $\sigma \in \mathcal{O}_\mathcal{A} = Act^*$ on the action display and $\alpha \in \mathcal{O}_\mathcal{T} = (\mathbb{R}_{\geq 0})^*$ on the time display of trace machine TM, (when the test starts in state s) is denoted by $p_{TM}(s, \sigma, \alpha)$. We relate states s, s' iff for all $(\sigma, \alpha) \in \mathcal{O}_{TM}$ the probability $p_{TM}(s, \sigma, \alpha)$ equals $p_{TM}(s', \sigma, \alpha)$, where $\mathcal{O}_{TM}$ is a suitable subset of $\mathcal{O}_\mathcal{A} \times O_\mathcal{T} = Act^* \times (\mathbb{R}_{\geq 0})^*$. In the sequel, elements of $O_\mathcal{A} = Act^*$ and $O_\mathcal{T} = (\mathbb{R}_{\geq 0})^*$ are denoted by the greek letters σ and α, respectively. The length of σ is given by $|\sigma|$ and equals the number of components in σ. In a similar way $|\alpha|$ is defined.

We now consider two variants of the Markovian trace machine that differ in the information shown at the time display:

(1) The time display shows the absolute time and the observer records one *time check* at an arbitrary time instant between the occurrence of two successive actions. We refer to the respective machine/testing scenario as AT.

(2) For each step the observer provides a value for a hidden countdown timer that starts at the time instant when an action is executed. For each transition the time display shows if this timer expired before the next action is performed or not, i.e. the observer knows if the time duration between the two successive actions is lower or equal to the value of the countdown timer. We refer to the respective machine/testing scenario as CT.

The information on the time display gives a single real value per transition. The set of observations is given by

$$\mathcal{O}_{AT} = \bigcup_{n\in\mathbb{N}} Act^n \times \mathbb{R}^{n+1}_{\geq 0} \text{ in case } AT \text{ and } \mathcal{O}_{CT} = \bigcup_{n\in\mathbb{N}} Act^n \times \mathbb{R}^{n}_{\geq 0} \text{ in case } CT.$$

Observing $(\sigma, \alpha) = (\langle a_1, a_2, \ldots, a_n\rangle, \langle t_1, t_2, \ldots, t_n, t_{n+1}\rangle) \in \mathcal{O}_{AT}$ in machine AT can be informally interpreted as follows. Let $y_m = \sum_{i=1}^{m} t_i$. σ represents the sequence of observed actions and for $1 \leq m \leq n$ action a_m is performed in the time interval $(y_m, y_{m+1}]$. The process has to remain in the initial state until time instant t_1 is over and in the state reached via a_n until time instant y_{n+1} is over.

Definition 4.1 [The equivalences $\equiv_{AT}$ and $\equiv_{CT}$] For $\sigma \in Act^n$, $n \geq 1$, $s \in S$ let $p_{AT}(s, \epsilon, t) = 1 - Expo(q(s), t)$ and

$$p_{AT}(s, \sigma, \alpha) = \sum_{\pi \in Path(s,\sigma)} dp(\pi) \cdot Conv(E'(\pi), \alpha)$$

be the probability to observe $(\sigma, \alpha) \in \mathcal{O}_{AT}$ if the process starts in state s. In case CT, we have that in each state the sojourn time does not exceed a certain time bound. We define $p_{CT}(s, \epsilon, \epsilon) = 1$ [3] and for $(\sigma, \alpha) \in \mathcal{O}_{CT}$ let

$$p_{CT}(s, \sigma, \alpha) = \sum_{\pi \in Path(s,\sigma)} dp(\pi) \cdot Expo(E(\pi), \alpha)$$

be the probability to observe (σ, α) in machine CT if the process starts in s. The corresponding equivalence relations are denoted by $\equiv_{CT}$ and $\equiv_{AT}$. □

Example 4.2 Consider the two ACTMCs in Figure 2. States are drawn as nodes and transitions as edges labelled with transition probabilities (transition rate divided by the respective exit rate) and actions. The exit rate of each state is given by the number inside its node. States with equal exit rates are shaded equally. The outgoing transitions of the lower states are omitted.

[3] We use (ϵ, ϵ) for the empty observation.

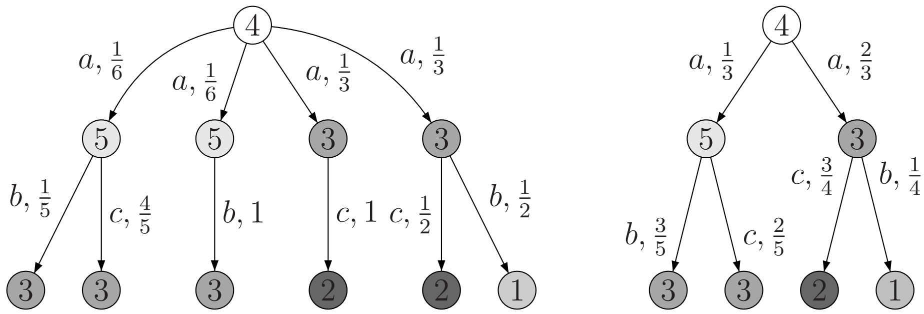

Fig. 2. Two trace equivalent ACTMCs.

Assume that the two states with exit rate 4 are s (left) and s' (right). Then, $s \equiv_{AT} s'$ and $s \equiv_{CT} s'$. For example, we have

$$\begin{aligned} p_{CT}(s, ab, \langle t, t' \rangle) &= \tfrac{1}{6} \cdot \tfrac{1}{5} \cdot Expo((4,5), \langle t, t' \rangle) + \tfrac{1}{6} \cdot 1 \cdot Expo((4,5), \langle t, t' \rangle) \\ &+ \tfrac{1}{3} \cdot \tfrac{1}{2} \cdot Expo((4,3), \langle t, t' \rangle) \\ &= \tfrac{1}{5} \cdot Expo((4,5), \langle t, t' \rangle) + \tfrac{1}{6} \cdot Expo((4,3), \langle t, t' \rangle) \\ &= \tfrac{1}{3} \cdot \tfrac{3}{5} \cdot Expo((4,5), \langle t, t' \rangle) + \tfrac{2}{3} \cdot \tfrac{1}{4} \cdot Expo((4,3), \langle t, t' \rangle) \\ &= p_{CT}(s', ab, \langle t, t' \rangle). \end{aligned}$$

□

We now present our main result stating that the two trace machines have equivalent distinguishing power and provide alternative characterizations of $\equiv_{AT} = \equiv_{CT}$ that abstract away from the concrete timed observations and just refer to the average sojourn times in the states.

Theorem 4.3 *For all states s_1, s_2 the following conditions are equivalent:*

(i) $s_1 \equiv_{CT} s_2$,

(ii) $s_1 \equiv_{AT} s_2$,

(iii) $\forall\, n \in \mathbb{N},\ \sigma \in Act^n, E \in \mathbb{R}^n_{\geq 0} : \sum_{\pi \in Path(s_1, \sigma, E)} dp(\pi) = \sum_{\pi \in Path(s_2, \sigma, E)} dp(\pi)$,

(vi) $\forall\, n \in \mathbb{N},\ \sigma \in Act^n, E' \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{\pi \in Path(s_1, \sigma, E')} dp(\pi) = \sum_{\pi \in Path(s_2, \sigma, E')} dp(\pi)$.

Before presenting the proof of Theorem 4.3 we explain how the alternative characterization of Markovian trace equivalence as stated in *(iii)* can serve for algorithmic purposes. Theorem 4.3 shows that the question whether two states of a finite-state ACTMC $M = (S, \longrightarrow, s_{init})$ are Markovian trace equivalent can be answered by transforming the ACTMC M into an action-labelled discrete-time Markov chain (ADTMC) $\hat{M}$ with the transition probabilities of M (the so-called embedded DTMC) where the exit rates are treated as additional edge labels.[4] Note that this transformation can be done in polynomial time. Then, states s_1, s_2 in M are Markovian trace equivalent if and only if s_1, s_2 are probabilistic trace equivalent in the embedded ADTMC $\hat{M}$ in the sense of [16]. The latter is decidable in polynomial-time as shown in [16]. Hence, we get:

Corollary 4.4 *If M is a finite-state ACTMC then the question whether $s_1 \equiv_{AT} s_2$ for two states s_1, s_2 in M is decidable in polynomial time.*

The remainder of this section is concerned with the proof of Theorem 4.3. We first establish a general result on exponential distributions that will be helpful in the following argumentation.

Proposition 4.5 *Let $q, q_1, \ldots, q_n \in \mathbb{R}_{>0}$, $q_i \neq q_j$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$, $d_1, \ldots, d_n \in \mathbb{R}$. Each of the following two conditions*

(1) $\sum_{j=1}^{n} Expo(q_j, t) \cdot d_j = 0$ *for all* $t \in \mathbb{R}_{\geq 0}$,

(2) $\sum_{j=1}^{n} d_j \cdot [expo(q, \cdot) * (1 - Expo(q_j, \cdot))](t) = 0$ *for all* $t \in \mathbb{R}_{\geq 0}$

implies $d_j = 0$ *for* $j = 1, ..., n$.

Proof. see appendix. □

The following transformation of an ACTMC $M = (S, \longrightarrow, s_{init})$ prepares the next lemma. For $s \in S, a \in Act, q \in \mathbb{R}_{>0}$ let

- $U(s, a, q) = \{s' \in S \mid r(s, a, s') > 0, q(s') = q\}$ be the set of direct a-successors of s with exit rate q,
- $Q(s, a) = \{q \in \mathbb{R}_{>0} \mid U(s, a, q) \neq \emptyset\}$ be the set of all exit rates of direct a-successors of s and
- $f(s, a, q) = p(s, a, U(s, a, q))$ the probability to reach a state with exit rate q via an a-transition from s.

The ACTMC $M^{(s,a)}$ is a copy of M where for each $q \in Q(s, a)$ new states $u^{(s,a,q)} \notin S$ are added. For all q the a-transitions from s to a state in $U(s, a, q)$ are replaced by one a-transition to $u^{(s,a,q)}$ with probability $f(s, a, q)$ and the outgoing transitions of $u^{(s,a,q)}$ are a copy of all outgoing transitions from states $u' \in U(s, a, q)$ weighted with $p(s, a, u')$.

[4] An ADTMC, called generative process in the classification of [28], with the action set $\hat{Act}$ is a triple $\hat{M} = (S, P, s_{init})$ where $P : S \times \hat{Act} \times S \rightarrow [0, 1]$ is a three-dimensional probability matrix such that $\sum_{s' \in S, \hat{a} \in \hat{Act}} P(s, \hat{a}, s') \in \{0, 1\}$ for all states $s \in S$. In our case, the underlying action set $\hat{Act}$ consists of pairs (a, q) where $a \in Act$ is an action in M and $q = q(s)$ is the exit rate of a state s in M.

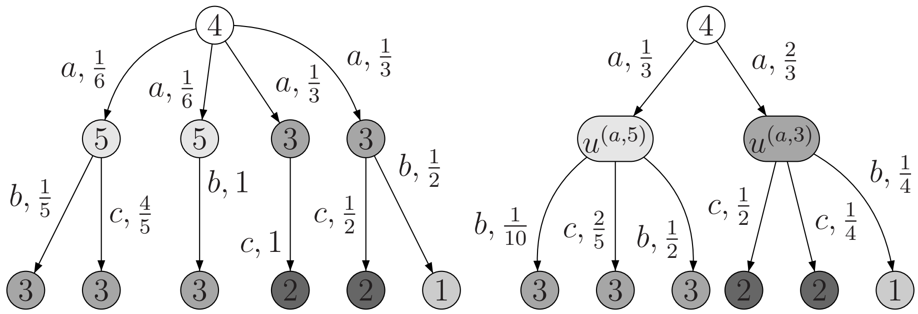

Fig. 3. ACTMC M and the transformed ACTMC $M^{(s,a)}$.

Definition 4.6 **[Transformed ACTMC]** Let $M = (S, \longrightarrow, s_{init})$ be an ACTMC and $s \in S, a \in Act$. The transformed ACTMC $M^{(s,a)} = (S \cup \{u^{(s,a,q)} \mid q \in Q(s,a)\}, \longrightarrow', s_{init})$ where $\longrightarrow'$ is such that $q(u^{(s,a,q)}) = q$ and

- for each transition $u' \xrightarrow{(b,p'\cdot q)} v$ of a state $u' \in U(s,a,q)$ we have $u^{(s,a,q)} \xrightarrow{(b,p\cdot q)}{}' v$ where $p = p' \cdot p(s,a,u')/f(s,a,q)$,
- $s' \xrightarrow{(b,r)}{}' s''$ iff $s' \xrightarrow{(b,r)} s''$ with $(s \neq s' \vee b \neq a)$,
- $s \xrightarrow{(a,p\cdot q(s))}{}' u^{(s,a,q)}$ with $p = f(s,a,q)$, $q \in Q(s,a)$.

□

Example 4.7 Figure 3 shows ACTMC M (left) of Example 4.2 starting in state s labelled with exit rate 4. The transformed ACTMC $M^{(s,a)}$ is shown on the right side. It holds that $Q(s,a) = \{3, 5\}$. The copies of direct successors of s in M are omitted and for the aggregated states $u^{(s,a,5)}$ and $u^{(s,a,3)}$ we simply write $u^{(a,5)}$ and $u^{(a,3)}$, respectively. We have, for instance, that $f(s,a,5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ and the transition probability for the left b-transition of $u^{(a,5)}$ is given by $\frac{1}{10} = \frac{1}{5} \cdot \frac{1}{6} \cdot \frac{3}{1}$.

Lemma 4.8 **[Characterization of $\equiv_{CT}$]** *$s_1 \equiv_{CT} s_2$ implies $f(s_1,a,q) = f(s_2,a,q)$ and $u^{(s_1,a,q)} \equiv_{CT} u^{(s_2,a,q)}$ for all $q \in Q(s,a)$.*

Proof. see appendix. □

Proof of Theorem 4.3.

$(iii) \Rightarrow (i)$: Assume $\sum_{\pi \in Path(s_1,\sigma,E)} dp(\pi) = \sum_{\pi \in Path(s_2,\sigma,E)} dp(\pi)$ and let $Q(s_i,\sigma) = \{E(\pi) | \pi \in Path(s_i,\sigma)\} \subset \mathbb{R}_{\geq 0}^{|\sigma|}$ be the set of all vectors of exit rates of paths from s_i with trace σ. Let $\alpha \in \mathbb{R}_{\geq 0}^{|\sigma|}$. Multiplying both sides with $Expo(E,\alpha)$ and summing up over all $E \in Q(s_1,\sigma) \cup Q(s_2,\sigma)$ yields

$$\sum_{E\in Q(s_1,\sigma)\cup Q(s_2,\sigma)} Expo(E,\alpha)\cdot\left[\sum_{\pi\in Path(s_1,\sigma,E)} dp(\pi) - \sum_{\pi\in Path(s_2,\sigma,E)} dp(\pi)\right] = 0$$

$$\iff \sum_{E\in Q(s_1,\sigma)} Expo(E,\alpha)\cdot \sum_{\pi\in Path(s_1,\sigma,E)} dp(\pi) = \sum_{E\in Q(s_2,\sigma)} Expo(E,\alpha)\cdot \sum_{\pi\in Path(s_2,\sigma,E)} dp(\pi)$$

$$\iff p_{CT}(s_1,\sigma,\alpha) = p_{CT}(s_2,\sigma,\alpha).$$

This holds for all $(\sigma,\alpha) \in Act^n \times \mathbb{R}^n_{\geq 0}$, $n \in \mathbb{N}$ and hence $s_1 \equiv_{CT} s_2$.

$(i) \Rightarrow (iii)$: This follows by Lemma 4.8: Assume that $E = (e_0, e_1, \ldots, e_{n-1}) \in \mathbb{R}^n_{\geq 0}$, $\sigma = a_1 \ldots a_n \in Act^n$, $e_0 = q(s_i)$ and M is transformed to $M(i,2) := M^{(s_i,a_1)}$. So $M(i,2)$ now contains the states $u(i,2) := u^{(s_i,a_1,e_1)}$ that represent the a_1-transitions from s_i to a state with exit rate e_1. Now, $M(i,2)$ is transformed to $M(i,3) := M(i,2)^{(u(i,2),a_2,e_2)}$ and so on. For $j = 2,\ldots,n-1$ we have $u(i,j+1) = u^{(u(i,j),a_j,e_j)}$ in ACTMC $M(i,j+1) = M(i,j)^{(u(i,j),a_j)}$. Then

$$\sum\nolimits_{\pi\in Path(s_i,\sigma,E)} dp(\pi) = f(s_i,a_1,e_1)\cdot f(u(i,2),a_2,e_2)\cdot\ldots \cdot f(u(i,n-1),a_{n-1},e_{n-1})\cdot p(u(i,n),a_n,S). \tag{1}$$

and for $j = 2,\ldots,n$ we have that $u(1,j) \equiv_{CT} u(2,j)$ implies

$$f(u(1,j),a_j,e_j) = f(u(2,j),a_j,e_j). \tag{2}$$

Furthermore, we have that $u(1,n) \equiv_{CT} u(2,n)$ implies $p(u(1,n),a_n,S) = p(u(2,n),a_n,S)$ and together with Equation 2 we obtain from 1

$$\sum_{\pi\in Path(s_1,\sigma,E)} dp(\pi) = \sum_{\pi\in Path(s_2,\sigma,E)} dp(\pi) \text{ for all } \sigma \in Act^n, E \in \mathbb{R}^n_{\geq 0}, n \in \mathbb{N}.$$

$(ii) \iff (iv)$: Goes along the same line as $(iii) \iff (i)$, but we use implication (2) instead of (1) in Proposition 4.5 and the fact that $s_1 \equiv_{AT} s_2$ implies $q(s_1) = q(s_2)$.

$(iii) \Rightarrow (iv)$: Let $E' \in \mathbb{R}^{n+1}_{\geq 0}$ and $\sigma \in Act^n$. Then if (iii) holds, we have

$$\sum_{\pi\in Path(s_1,\sigma,E')} dp(\pi) = \sum_{a\in Act}\sum_{\pi\in Path(s_1,\sigma a,E')} dp(\pi) = \sum_{a\in Act}\sum_{\pi\in Path(s_2,\sigma a,E')} dp(\pi) = \sum_{\pi\in Path(s_2,\sigma,E')} dp(\pi).$$

$(iv) \Rightarrow (iii)$: Follows directly if we sum up over all possible last components of E' in (iv).

□

5 Variants of Markovian Trace Equivalence

Completed traces. In the non-probabilistic setting a completed trace is a trace that ends up in a deadlock state and a trace equivalence based on a distinction of completed and "incompleted" traces is strictly finer than the standard non-probabilistic trace equivalence [27]. For the trace machine this means that the action display shows the current action as usual and if a deadlock state is reached, a special symbol $\perp \notin Act$ appears in the action display. In the probabilistic setting this does not, as opposed to the non-probabilistic case, increase distinguishing power. This was shown for the time-abstract case by Huynh and Tian [16] and stems from the fact that the (discrete-time) probability of a completed trace σ can be expressed in terms of the probability of a not necessarily completed trace σ and the trace σ' with $\sigma' = \sigma a$ for some $a \in Act$. It is obvious that in the continuous time setting the same holds, i.e. completed trace equivalence on ACTMCs coincides with trace equivalence on ACTMCs.

Failure and ready semantics analyse the deadlock and liveness behaviour of processes and in the non-probabilistic setting the associated equivalences are strictly finer than trace equivalence.

Failure traces. A *failure set* of a state s in an ACTMC is a set of actions that can not be carried out from s, i.e. $A \subseteq Act$ is a failure set if $p(s, a, S) = 0$ for all $a \in Act$. An appropriate testing scenario for failure semantics consists of a trace machine with reset button, action and time display as for the case CT (compare Section 4) but for each $a \in Act$ there is an action light in addition such that in each step the action lights of a failure set flash. A non-probabilistic version of this scenario was originally described in [27]. An observation of length n is given by a triple (σ, α, ν) where

- $\sigma \in Act^n$ is the sequence of actions shown at the action display,
- $\alpha \in \mathbb{R}^n_{\geq 0}$ are the values of the countdown timer in each step,
- $\nu = \langle A_1, A_2, \ldots, A_n \rangle, A_i \subseteq Act, 1 \leq i \leq n$ describes the failure sets observed in each step.

A sequence $\nu = \langle A_1, A_2, \ldots, A_n \rangle$ is a *failure trace* of path $\pi = \langle s_0, b_1, s_1, b_2, \ldots, s_{n-1}, b_n, s_n \rangle$ if $p(s_i, a_{i+1}, S) = 0$ for all $a_{i+1} \in A_{i+1}$, $0 \leq i < n$. Note that $b_i \notin A_i$ since π is a path. Let $ftrace(\pi)$ denote the set of failure traces of π.

The probability of an observation (σ, α, ν) if the ACTMC starts in state s is given by

$$p_{FT}(s, \sigma, \alpha, \nu) = \sum_{\pi \in Path_f(s,\sigma,\nu)} dp(\pi) \cdot Expo(E(\pi), \alpha),$$

where $Path_f(s, \sigma, \nu) = \{\pi \in Path(s, \sigma) \mid \nu \in ftrace(\pi)\}$. Let $\nu^{\emptyset} = \langle \emptyset \ldots \emptyset \rangle$.

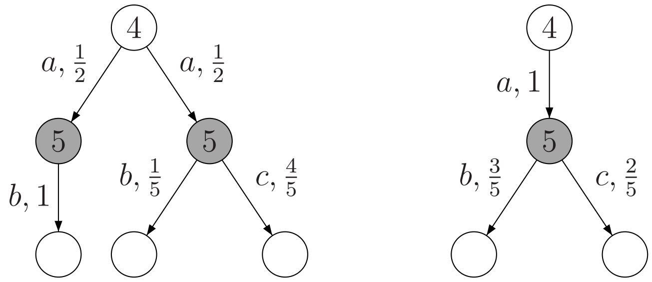

Fig. 4. $s \equiv_{CT} s'$, but $s \not\equiv_{FT} s'$.

Then

$$p_{FT}(s, \sigma, \alpha, \nu^{\emptyset}) = p_{CT}(s, \sigma, \alpha). \tag{3}$$

Let $\equiv_{FT}$ denote the relation that distinguishes states according to the testing scenario for failure traces, i.e. $s \equiv_{FT} s'$ iff

$$p_{FT}(s, \sigma, \alpha, \nu) = p_{FT}(s', \sigma, \alpha, \nu) \; \forall (\sigma, \alpha, \nu) \in Act^n \times \mathbb{R}^n_{\geq 0} \times (2^{Act})^n, n \in \mathbb{N}.$$

From Equation 3 we derive directly that $\equiv_{FT} \subseteq \equiv_{CT}$. The following counter example shows that $\equiv_{FT}$ is strictly finer than $\equiv_{CT}$, i.e. $\equiv_{CT} \not\subseteq \equiv_{FT}$.

Example 5.1 Consider the two ACTMCs in Figure 4 starting in the two upper states s and s' with exit rate 4, respectively. It is easy to see that $s \equiv_{CT} s'$, but for observation $(\sigma, \alpha, \nu) = (ab, \langle t_1, t_2 \rangle, \langle \{\}, \{c\} \rangle)$ in scenario FT we have $p_{FT}(s, \sigma, \alpha, \nu) = \frac{1}{2} \cdot 1 \cdot Expo(4, 5, \langle t_1, t_2 \rangle) > 0 = p_{FT}(s', \sigma, \alpha, \nu)$. □

In [16] failure traces for the discrete-time case are analysed that consider failure sets only for the last state of a trace. This is a special case of the failure traces here and can be analysed by restricting the observations of the action lights to sequences $\langle A_1, A_2, \ldots, A_n \rangle$ with $A_i = \emptyset$ for $1 \leq i < n$. It is easy to show that the resulting equivalence is strictly finer than $\equiv_{CT}$ but strictly coarser than $\equiv_{FT}$ just as in the non-probabilistic setting. A detailed analysis of this scenario is omitted here.

Ready traces. The *ready set* of a state s in an ACTMC is the set of actions the process can perform with a non-zero probability from s. For path $\pi = \langle s_0, a_1, s_1, a_2, \ldots, s_{n-1}, a_n, s_n \rangle$ we define $rtrace(\pi) = \langle A_1, A_2, \ldots, A_n \rangle$ where $A_{i+1} = \{a \in Act \mid p(s_i, a, S) > 0\}$, $0 \leq i < n$. Note that $A_i \neq \emptyset$ because $a_i \in A_i$. The ready trace machine is similar to the failure trace machine, but the action lights show the ready set in each step instead of a failure set. The probability of observing (σ, α, ν) in the ready trace machine is given by

$$p_{RT}(s, \sigma, \alpha, \nu) = \sum_{\pi \in Path_r(s, \sigma, \nu)} dp(\pi) \cdot Expo(E(\pi), \alpha),$$

where $Path_r(s, \sigma, \nu) = \{\pi \in Path(s, \sigma) \mid rtrace(\pi) = \nu\}$. Let $\equiv_{RT}$ denote the resulting equivalence. It holds that

$$p_{CT}(s,\sigma,\alpha) = \sum_{\nu \in (2^{Act})^n} p_{RT}(s,\sigma,\alpha,\nu),$$

where $\sigma \in Act^n, \alpha \in \mathbb{R}^n_{\geq 0}$. This implies that $\equiv_{RT} \subseteq \equiv_{CT}$. The opposite does not hold since $\equiv_{RT}$ coincides with $\equiv_{FT}$ (and $\equiv_{FT}$ is strictly finer than $\equiv_{CT}$):

Proposition 5.2 *For states s_1, s_2 of an ACTMC: $s_1 \equiv_{RT} s_2$ iff $s_1 \equiv_{FT} s_2$.*

Proof. Let $(\sigma,\alpha,\nu) \in Act^n \times \mathbb{R}^n_{\geq 0} \times (2^{Act})^n$. Then

(1) $p_{FT}(s,\sigma,\alpha,\nu) = \sum_{\nu' \in (2^{Act})^n : \nu \cap \nu' = \emptyset} p_{RT}(s,\sigma,\alpha,\nu')$,

(2) $p_{RT}(s,\sigma,\alpha,\nu) = p_{FT}(s,\sigma,\alpha, Act \setminus \nu) - \sum_{\nu' \subsetneq \nu} p_{FT}(s,\sigma,\alpha,\nu')$,

where for $\nu = \langle A_1, A_2, \ldots, A_n \rangle$, $\nu' = \langle A'_1, A'_2, \ldots, A'_n \rangle$ we write $\nu \cap \nu' = \emptyset$ if $A_i \cap A'_i = \emptyset$, $1 \leq i \leq n$, $Act \setminus \nu$ denotes the sequence $\langle Act \setminus A_1, Act \setminus A_2, \ldots, Act \setminus A_n \rangle$ and $\nu' \subsetneq \nu$ if $A'_i \subsetneq A_i$ for all $1 \leq i \leq n$. □

In [16] a similar relation between failure and ready semantics was established for discrete-time Markov chains, but for the case that only the ready/failure set of the last state of the trace is considered.

A characterization of failure/ready trace equivalence as in part (*iii*) of Theorem 4.3 which refers to the exit rates rather than to the observable timing behaviour could be given as well which – in combination with the results of [16] – yields that failure/ready trace equivalence are decidable for finite-state ACTMCs in polynomial time.

6 Conclusion

We studied several linear-time equivalences for continuous-time Markov chains and concentrated here on testing scenarios for them. Other properties of interest (congruence properties, logical characterizations, axiomatizations, etc.) will be investigated in future work. The relations considered here are all strictly coarser than bisimulation equivalence for ACTMCs. The precise relation to Markovian testing equivalence defined in [1] is still unclear. Beside the differences between our machine-based framework and the testing approach of [1] that we mentioned in the introduction, the equivalence defined in [1] relies on observing an action sequence and the total amount of time needed to perform this action sequence while our equivalence takes the intermediate time passage between the actions into account. Another difference between the relations defined in [1] and our approach is that in [1] the observer has the possibility to "block" certain actions such that the tested process may only execute unblocked actions. Although a characterization of the equivalence in

[1] is possible in terms of an appropriate trace machine as in Fig. 1, the precise connection to the relations considered here is not yet clear to us.

Our future directions include the investigation of the linear - branching time spectrum for continuous-time models with nondeterminism, such as interactive Markov chains [13].

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A Proof of Lemma 4.8 and Proposition 4.5

We first provide the proof for Lemma 4.8 stating that $s_1 \equiv_{CT} s_2$ implies $f(s_1,a,q) = f(s_2,a,q)$ and $u^{(s_1,a,q)} \equiv_{CT} u^{(s_2,a,q)}$ for all $q \in Q(s,a)$.

Proof. Assume that $s_1 \equiv_{CT} s_2$. First consider observation $(a,t) \in Act \times \mathbb{R}_{\geq 0}$. Then $q(s_1) = q(s_2) := q_{1,2}$, because we have for all $t \in \mathbb{R}_{\geq 0}$

$$\sum_{a \in Act} p_{CT}(s_1,a,t) = \sum_{a \in Act} p_{CT}(s_2,a,t) \iff Expo(q(s_1),t) = Expo(q(s_2),t).$$

Now consider observation $(ab, \langle t,t' \rangle)$. Let $i \in \{1,2\}$. It holds that

$$\begin{aligned}\sum_{b \in Act} p_{CT}(s_i,ab,\langle t,t' \rangle) &= \sum_{b \in Act} \sum_{\pi \in Path(s_i,ab)} Expo(e(\pi),\langle t,t' \rangle) \cdot dp(\pi) \\ &= Expo(q_{1,2},t) \cdot \sum_{s' \in S} Expo(q(s'),t') \cdot p(s_i,a,s').\end{aligned}$$

Since $\sum_{b \in Act} p_{CT}(s_1,ab,\langle t,t' \rangle) = \sum_{b \in Act} p_{CT}(s_2,ab,\langle t,t' \rangle)$ we derive

$$\begin{aligned}&\tfrac{1}{Expo(q_{1,2},t)} \cdot \left[\sum_{b \in Act} p_{CT}(s_1,ab,\langle t,t' \rangle) - \sum_{b \in Act} p_{CT}(s_2,ab,\langle t,t' \rangle)\right] = 0 \iff \\ &\sum_{s' \in S} Expo(q(s'),t') \cdot p(s_1,a,s') - \sum_{s' \in S} Expo(q(s'),t') \cdot p(s_2,a,s') = 0\end{aligned}$$

Hence for all $t' \in \mathbb{R}_{\geq 0}$

$$\sum_{\substack{q \in Q(s_1,a) \\ \cup Q(s_2,a)}} Expo(q,t') \cdot \left[\sum_{u \in U(s_1,a,q)} p(s_1,a,u) - \sum_{u \in U(s_2,a,q)} p(s_2,a,u)\right] = 0.$$

We use Proposition 4.5 and get for all $q \in Q(s_1,a) \cup Q(s_2,a)$:

$$f(s_1,a,q) = \sum_{u \in U(s_1,a,q)} p(s_1,a,u) = \sum_{u \in U(s_2,a,q)} p(s_2,a,u) = f(s_2,a,q).$$

Note that $Q(s_1,a) = Q(s_2,a)$ follows directly for all $a \in Act$.
Now we show that $u^{(s_1,a,q)} \equiv_{CT} u^{(s_2,a,q)}$, i.e. for all observations (σ,α)

$$p_{CT}(u^{(s_1,a,q)},\sigma,\alpha) = p_{CT}(u^{(s_2,a,q)},\sigma,\alpha).$$

It holds that for all $a,b \in Act$, $t,t' \in \mathbb{R}_{\geq 0}$, $(\sigma,\alpha) \in Act^n \times \mathbb{R}^n_{\geq 0}$, $n \geq 0$

$$p_{CT}(s_1,ab\sigma,\langle t,t',\alpha \rangle) = p_{CT}(s_2,ab\sigma,\langle t,t',\alpha \rangle).$$

With $q(s_1) = q(s_2)$ and a similar argument as above we derive

$$\sum_{q \in Q(s_i,a)} Expo(q,t') \cdot \Bigg[\sum_{u \in U(s_1,a,q)} p(s_1,a,u) \sum_{v \in S} p(u,b,v) \cdot p_{CT}(v,\sigma,\alpha) - \sum_{u \in U(s_2,a,q)} p(s_2,a,u) \sum_{v \in S} p(u,b,v) \cdot p_{CT}(v,\sigma,\alpha) \Bigg] = 0.$$

Again we use Proposition 4.5, so for all $q \in Q(s_i,a)$:

$$\sum_{u \in U(s_1,a,q)} p(s_1,a,u) \sum_{v \in S} p(u,b,v) \cdot p_{CT}(v,\sigma,\alpha) = \sum_{u \in U(s_2,a,q)} p(s_2,a,u) \sum_{v \in S} p(u,b,v) \cdot p_{CT}(v,\sigma,\alpha).$$

We multiply both sides with $Expo(q,t')/f(s_i,a,q)$ and since

$$Expo(q,t')/f(s_i,a,q) \cdot \sum_{u \in U(s_i,a,q)} p(s_i,a,u) \sum_{v \in S} p(u,b,v) \cdot p_{CT}(v,\sigma,\alpha) = p_{CT}(u^{(s_i,a,q)}, b\sigma, \langle t',\alpha \rangle)$$

we derive $p_{CT}(u^{(s_1,a,q)}, b\sigma, \langle t',\alpha\rangle) = p_{CT}(u^{(s_2,a,q)}, b\sigma, \langle t',\alpha\rangle)$ for all $b \in Act$, $t' \in \mathbb{R}_{\geq 0}$, $(\sigma,\alpha) \in Act^n \times \mathbb{R}^n_{\geq 0}$, $n \geq 0$. Hence $u^{(s_1,a,q)} \equiv_{CT} u^{(s_2,a,q)}$. □

We now turn to the proof of Proposition 4.5 which states that for all $q, q_1, \ldots, q_n \in \mathbb{R}_{>0}$, $q_i \neq q_j$ for $i \neq j$, $i,j \in \{1,\ldots,n\}$, $d_1,\ldots,d_n \in \mathbb{R}$, each of the following two conditions

(1) $\sum_{j=1}^{n} Expo(q_j,t) \cdot d_j = 0$ for all $t \in \mathbb{R}_{\geq 0}$,

(2) $\sum_{j=1}^{n} d_j \cdot [expo(q,\cdot) * (1 - Expo(q_j,\cdot))](t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$.

implies $d_j = 0$ for $j = 1, ..., n$.

Proof. Consider Equation (1). The i-th moment of a random variable exponentially distributed with parameter $q > 0$ exists and is defined by

$$M^{(i)}(q) = \int_{-\infty}^{\infty} x^i \cdot expo(q,x)dx = i! \cdot \frac{1}{q^i}.$$

For details about moments of a random variables see [8]. Thus taking the i-th moment on both sides of equation (1) and using the fact that this is a linear operator, we derive

$$\sum_{j=1}^{n} i! \cdot \frac{1}{q_j^i} \cdot d_j = 0 \implies \sum_{j=1}^{n} \frac{1}{q_j^i} \cdot d_j = 0, \forall i \in \{1,\ldots,n\}.$$

Let V be the matrix with entries $V_{i,j} = \frac{1}{q_j^i}$, $d = (d_1, \ldots, d_n)^T$ be a column vector with entries d_j and $\mathbf{0}$ be the column vector with zero entries. Then we can reformulate the above equations as the linear equation system $V \cdot d = \mathbf{0}$. Since V is a Vandermonde matrix, it is easy to derive that the determinant of V is $det(V) = \prod_{1 \leq i < j \leq n} (\frac{1}{q_j} - \frac{1}{q_i}) \neq 0$. Hence $d = \mathbf{0}$ is the unique solution of this equation system.
Case (2) goes along the same lines as case (1) except that $expo(q, \cdot)$ produces a constant factor that is canceled for each summand. □