Weighted Tree Automata

Exercise 24 (Closure properties)
Let $\Sigma$ be a r.a. We consider a very simple class of tree languages, called the local tree languages over $\Sigma$. First of all, we define the set of forks of $\Sigma$ as

$$\text{fork}(\Sigma) = \{ (\sigma, \sigma_1, \ldots, \sigma_k) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, \sigma_1, \ldots, \sigma_k \in \Sigma \}.$$ 

Let $G \subseteq \text{fork}(\Sigma)$ and $H \subseteq \Sigma$. The language defined by $(G,H)$ is

$$L(G,H) = \{ \xi \in T_{\Sigma} \mid \forall w \in \text{pos}(\xi) : \text{rk}(\xi(w)) > 0 \Rightarrow (\xi(w), \xi(w_1) \cdots \xi(w \text{rk}(\xi(w)))) \in G \land \xi(\epsilon) \in H \}.$$ 

We call a language $L \subseteq T_{\Sigma}$ a local tree language (over $\Sigma$) if there are $G, H$ such that $L = L(G,H)$. The class of all local tree languages over $\Sigma$ is denoted by Loc$(\Sigma)$.

Is Loc$(\Sigma)$ closed under union and tree concatenation? Extra task: What about the other operations we investigated for Rec$(\Sigma)$?

Exercise 25 (Construction of Bar-Hillel, Perles and Shamir)
Consider the r.a. $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}, \lambda^{(0)}\}$ and the FTA $A = (Q, \Sigma, \delta, F)$ where $Q = \{e, o\}$, $F = \{e\}$, and

$$\delta_\alpha = \delta_\beta = \delta_\gamma = \{(\epsilon, o)\} \quad \delta_\sigma = \{(q_1q_2, q_0) \in Q^2 \times Q \mid q_0 = o \text{ iff } q_1 = q_2\}.$$ 

Moreover, let us assume an FSA $B = (P, \Delta, p, \mu, G)$ where $\Delta = \Sigma^{(0)} \setminus \{\lambda\}$, $P = \{p, r\}$, $G = \{r\}$ and

$$\mu = \{(p, \alpha, p), (p, \beta, r), (r, \beta, r)\}.$$ 

Using the technique from the lecture, construct an FTA $A'$ such that $L(A') = L(A) \cap \text{yield}_\lambda^{-1}(L(B))$. Are there "superfluous" transitions in $A'$ which we might remove?