Task 7 (closure of Rec under intersection, union, and complement)

Let $\Sigma = \{\sigma(2), \gamma(1), \alpha(0), \beta(0)\}$ be a ranked alphabet. Consider the following recognizable tree languages

$L_1 = \{\xi \in T_\Sigma \mid \text{for every } w \in \text{pos}(\xi): w \in \{2\}^* \text{ if and only if } \xi(w) \in \{\sigma, \alpha\}\}$

$L_2 = \{\xi \in T_\Sigma \mid \text{for every } w \in \text{pos}(\xi): \xi(w) = \alpha \text{ only if } |w| \equiv 0 \pmod{2}\}$

Find finite representations for the following languages:

(a) $L_1$
(b) $L_2$
(c) $L_1 \cup L_2$
(d) $L_1 \cap L_2$
(e) $T_\Sigma \setminus L_1$

Task 8 (concatenation and Kleene star for recognizable tree languages)

Let $\Sigma$ be a ranked alphabet.

(a) Show that $\text{Rec}(\Sigma)$ is closed under top concatenation without using the fact that it is closed under tree concatenation.

(b) Why can we not use the closure of $\text{Rec}(\Sigma)$ under tree concatenation to prove the closure under Kleene star?

Prove or refute the following two statements:

(c) For every $\alpha \in \Sigma^{(0)}$, the binary operation $\cdot_\alpha$ is associative.

(d) $(L_1 \cdot_\alpha L_2) \cdot_\beta L_3 = L_1 \cdot_\alpha(L_2 \cdot_\beta L_3)$ for arbitrary $L_1, L_2, L_3 \in \text{Rec}(\Sigma)$ and $\alpha, \beta \in \Sigma^{(0)}$.

Let $\Delta = \{\sigma(2), \alpha(0), \beta(0)\}$ be a ranked alphabet.

(e) Using the construction from the lecture, show that $\{\sigma(\alpha, \beta)\}^*_\beta \cdot_\beta \{\alpha\} \in \text{Rec}(\Sigma)$.

Task 9 (finite state automata)

Let $\Sigma = \{a, b\}$ be an alphabet.

(a) Give a finite state automaton $A = (Q, \Sigma, q_0, F)$ that recognizes

$L = \{w \in \Sigma^* \mid |w|_a - |w|_b \mod{2} \equiv 0\}$

(b) Describe $L$ using a homomorphism between the free monoid $(\Sigma^*, \circ, \varepsilon)$ and the monoid $\langle \{0,1\}^Q \times Q, \times, 1_{Q \times Q} \rangle$.

(c) Describe $L$ using a monoid with carrier $(\Sigma^*)^{Q \times Q}$.