Chapter 4

Declarative Interpretation
Outline

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models
What is an Interpretation?

direct(frankfurt,san_francisco).
direct(frankfurt,chicago).
direct(san_francisco,honolulu).
direct(honolulu,maui).

collection(X, Y) :- direct(X, Y).
collection(X, Y) :- direct(X, Z), collection(Z, Y).

\[ D = \{ FRA, DRS, ORD, SFO, \ldots \} \]
frankfurt\(_j\) = FRA, chicago\(_j\) = ORD, san\_francisco\(_j\) = SFO, ...
direct\(_i\) = \{(FRA, SFO), (FRA, ORD), \ldots \}
collection\(_i\) = \{(FRA, SFO), (FRA, ORD), (FRA, HNL), \ldots \}\]
What is an Interpretation?

\text{add}(X, 0, X).
\text{add}(X, \text{s}(Y), \text{s}(Z)) :- \text{add}(X, Y, Z).

\(D = \mathbb{N}\)
\(0_J = 0\)
\(s_J : \mathbb{N} \rightarrow \mathbb{N} \text{ such that } s_J(n) = n + 1\)
\(\text{add}_I = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), \ldots\}\)
Another Example

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add(X, 0, X).
add(X, s(Y), s(Z)) :- add(X, Y, Z).
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\[D = \{0, s(0), s(s(0)), \ldots\}\]

\[0_j = 0\]

\[s_j : D \rightarrow D \text{ such that } s_j(t) = s(t)\]

\[\text{add}_i = \{(0, 0, 0), (s(0), 0, s(0)), (0, s(0), s(0)), (s(0), s(0), s(s(0))), \ldots\}\]

(This will be called a “Herbrand model”.)
Algebras

$V$ set of variables, $F$ ranked alphabet of function symbols: An algebra $J$ for $F$ (or pre-interpretation for $F$) consists of:

1. domain $\Leftrightarrow$ non-empty set $D$
2. assignment of a mapping

   $f_J : D^n \rightarrow D$

   to every $f \in F^{(n)}$ with $n \geq 0$

**State** $\sigma$ over $D$ $\Leftrightarrow$ mapping $\sigma : V \rightarrow D$

**Extension of** $\sigma$ to $TU_{F,V}$ $\Leftrightarrow$ $\sigma : TU_{F,V} \rightarrow D$ such that for every $f \in F^{(n)}$

$$\sigma(f(t_1, \ldots, t_n)) = f_J(\sigma(t_1), \ldots, \sigma(t_n))$$
Interpretations

$F$ ranked alphabet of function symbols, $\Pi$ ranked alphabet of predicate symbols:

An interpretation $I$ for $F$ and $\Pi$ consists of:

1. algebra $J$ for $F$ (with domain $D$)
2. assignment of a relation

$$ p_I \subseteq \underbrace{D \times \ldots \times D}_n $$

... to every $p \in \Pi^{(n)}$ with $n \geq 0$
Herbrand Universes and Bases

Recall $TU_{F,V} \iff$ term universe over function symbols $F$, variables $V$

$TB_{\Pi,F,V} \iff$ term base (i.e., all atoms) over predicate symbols $\Pi$ and $F, V$

- Herbrand universe $HU_{F} \iff TU_{F,\emptyset}$

- Herbrand base $HB_{\Pi,F} \iff TB_{\Pi,F,\emptyset}$
Interpretations (Example)

Let $P_{\text{add}}$ “add-program”.

$P_{\text{add}}$ is an extended program of the form

\begin{align*}
& \text{add} \quad \text{add-copy} \\
& \text{add}(\mathit{add}(0)) \quad \text{add}(\mathit{add}(1)) \quad \text{add}(\mathit{add}(s(0))) \\
& \text{add}(\mathit{add}(s(1))) \\
& \text{add}(\mathit{add}(s(s(0)))) \\
& \text{add}(\mathit{add}(s(s(1)))) \\
& \text{add}(\mathit{add}(s(s(s(0))))) \\
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& \text{add}(\mathit{add}(s(s(s(s(s(s(1)))))))) \\
& \text{add}(\mathit{add}(s(s(s(s(s(s(s(0))))))))))
\end{align*}

Let $I_1, I_2, I_3, I_4, I_5, \text{ and } I_6$ be interpretations for $\{\mathit{s}, 0\}$ and $\{\text{add}\}$:

$I_1$: $D_{I_1} = \mathbb{N}$, $0_{I_1} = 0$, $s_{I_1}(n) = n + 1$ for each $n \in \mathbb{N}$, $\text{add}_{I_1} = \{(m, n, m + n) \mid m, n \in \mathbb{N}\}$

$I_2$: $D_{I_2} = \mathbb{N}$, $0_{I_2} = 0$, $s_{I_2}(n) = n + 1$ for each $n \in \mathbb{N}$, $\text{add}_{I_2} = \{(m, n, m * n) \mid m, n \in \mathbb{N}\}$

$I_3$: $D_{I_3} = \text{HU}_{\{\mathit{s}, 0\}}$, $0_{I_3} = 0$, $s_{I_3}(t) = s(t)$ for each $t \in \text{HU}_{\{\mathit{s}, 0\}}$, $\text{add}_{I_3} = \{(s^m(0), s^n(0), s^{m+n}(0)) \mid m, n \in \mathbb{N}\}$

$I_4$: $D_{I_4} = \text{HU}_{\{\mathit{s}, 0\}}$, $0_{I_4} = 0$, $s_{I_4}(t) = s(t)$ for each $t \in \text{HU}_{\{\mathit{s}, 0\}}$, $\text{add}_{I_4} = \emptyset$

$I_5$: $D_{I_5} = \text{HU}_{\{\mathit{s}, 0\}}$, $0_{I_5} = 0$, $s_{I_5}(t) = s(t)$ for each $t \in \text{HU}_{\{\mathit{s}, 0\}}$, $\text{add}_{I_5} = \left(\text{HU}_{\{\mathit{s}, 0\}}\right)^3$

$I_6$: $D_{I_6} = \{0, 1\}$, $0_{I_6} = 0$, $s_{I_6}(n) = n$ for each $n \in \{0, 1\}$, $\text{add}_{I_6} = \{(m, n, m) \mid m, n \in \{0, 1\}\}$
Logical Truth (I)

$E$ expression $\iff$ $E$ atom, query, clause, or resultant

$E$ expression, $I$ interpretation, $\sigma$ state:

$E$ true in $I$ under $\sigma$, written: $I \models_\sigma E$

$\iff$

by case analysis on $E$:

- $I \models_\sigma p(t_1, \ldots, t_n) \iff (\sigma(t_1), \ldots, \sigma(t_n)) \in p_I$
- $I \models_\sigma A_1, \ldots, A_n \iff I \models_\sigma A_i$ for every $i = 1, \ldots, n$
- $I \models_\sigma A \leftarrow B \iff$ if $I \models_\sigma B$ then $I \models_\sigma A$
- $I \models_\sigma A \leftarrow B \iff$ if $I \models_\sigma B$ then $I \models_\sigma A$
Logical Truth (II)

$E$ expression, $I$ interpretation:
Let $x_1, \ldots, x_k$ be the variables occurring in $E$.

- $\forall x_1, \ldots, \forall x_k \ E$ universal closure of $E$ (abbreviated $\forall E$)
- $\exists x_1, \ldots, \exists x_k \ E$ existential closure of $E$ (abbreviated $\exists E$)
- $I \models \forall E \iff I \models_{\sigma} E$ for every state $\sigma$
- $I \models \exists E \iff I \models_{\sigma} E$ for some state $\sigma$
- $E$ true in $I$ (or: $I$ model of $E$), written: $I \models E \iff I \models \forall E$
Logical Truth (III)

S, T sets of expressions, I interpretation:

- I model of S, written: $I \models S \iff I \models E$ for every $E \in S$
- T semantic (or: logical) consequence of S, written $S \models T$
  : $\iff$ every model of S is a model of T

$P$ program, $Q_0$ query, $\theta$ substitution:

- $\theta \models_{\text{Var}(Q_0)}$ correct answer substitution of $Q_0$ : $\iff P \models Q_0\theta$
- $Q_0\theta$ correct instance of $Q_0$ : $\iff P \models Q_0\theta$
Models (Example)

Let $P_{\text{add}}$ “add-program” and let $l_1, l_2, l_3, l_4, l_5,$ and $l_6$ be the interpretations from slide 9.

- $l_1 \models P_{\text{add}}$ (since $l_1 \models_\sigma c$ for every clause $c \in P_{\text{add}}$ and state $\sigma : V \rightarrow \mathbb{N}$:
  (i) $(\sigma(x), \sigma(0), \sigma(x)) \in \text{add}_{l_1}$ and
  (ii) if $(\sigma(x), \sigma(y), \sigma(z)) \in \text{add}_{l_1}$ then $(\sigma(x), \sigma(y)+1, \sigma(z)+1) \in \text{add}_{l_1}$

- $l_2 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = 1$, then $l_2 \not\models_\sigma \text{add}(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (1, 0, 1) \not\in \text{add}_{l_2}$)

- $l_3 \models P_{\text{add}}$ (like for $l_1$; we call $l_3$ a (least) Herbrand model)

- $l_4 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = s(0)$, then $l_4 \not\models_\sigma \text{add}(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (s(0), 0, s(0)) \not\in \text{add}_{l_4}$)

- $l_5 \models P_{\text{add}}$ (like for $l_1$; we call $l_5$ a Herbrand model)

- $l_6 \models P_{\text{add}}$ (like for $l_1$)
Semantic Consequences (Example)

Let $P_{add}$ “add-program”.

- $P_{add} \models add(x, 0, x)$
  (for every interpretation $I$ : if $I \models P_{add}$ then $I \models add(x, 0, x)$, since $add(x, 0, x) \in P_{add}$)

- $P_{add} \models add(x, s(0), s(x))$
  (for every interpretation $I$ : if $I \models P_{add}$ then $I \models add(x, 0, x)$
  and $I \models add(x, s(0), s(x)) \leftarrow add(x, 0, x)$ (instance of clause), thus $I \models add(x, s(0), s(x))$)

- $P_{add} \not\models add(0, x, x)$
  (consider interpretation $I_6$ from slide 9 with $I_6 \models P_{add}$: $I_6 \not\models add(0, x, x)$, since e.g. $I_6 \not\models_{\sigma} add(0, x, x)$ for $\sigma(x) = 1$,
  since $(\sigma(0), \sigma(x), \sigma(x)) = (0, 1, 1) \notin add_{I_6}$)
Lemma 4.3 (i)

Let \( Q \implies Q' \) be an SLD-derivation step and \( Q\theta \leftarrow Q' \) the resultant associated with it. Then

\[ c \models Q\theta \leftarrow Q' \]

Proof.

Let \( Q = A, B, C \) with selected atom \( B \). Let \( H \leftarrow B \) be the input clause and \( Q' = (A, B, C)\theta \). Then

\[ c \models H \leftarrow B \] (variant of \( c \))

implies \[ c \models H\theta \leftarrow B\theta \] (instance)

implies \[ c \models B\theta \leftarrow B\theta \] (\( \theta \) unifier)

implies \[ c \models (A, B, C)\theta \leftarrow (A, B, C)\theta \] (“context” unchanged)
Towards Soundness of SLD-Resolution (II)

Lemma 4.3 (ii)

Let $\xi$ be an SLD-derivation of $P \cup \{Q_0\}$. For $i \geq 0$ let $R_i$ be the resultant of level $i$ of $\xi$. Then $P \models R_i$

Proof.

Let $\xi = Q_0 \Rightarrow Q_1 \Rightarrow \ldots \Rightarrow Q_n \Rightarrow Q_{n+1} \Rightarrow \ldots$. Induction on $i \geq 0$:

$i = 0$: $R_0 = Q_0 \leftarrow Q_0 = \text{"true"}$, thus $P \models R_0$

$i = 1$: $R_1 = Q_0 \theta_1 \leftarrow Q_1$; by Lemma 4.3 (i): $P \models R_1$

$i \Rightarrow i + 1$: $R_{i+1} = Q_0 \theta_1 \ldots \theta_{i+1} \leftarrow Q_{i+1}$ is a semantic consequence of resultant $Q_i \theta_{i+1} \leftarrow Q_{i+1}$ associated with $(i + 1)$-st derivation step and $R_{i+1} = Q_0 \theta_1 \ldots \theta_{i+1} \leftarrow Q_i \theta_{i+1}$, thus by Lemma 4.3 (i) and induction hypothesis: $P \models R_{i+1}$
Soundness of SLD-Resolution

Theorem 4.4

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$ with CAS $\theta$, then $P \models Q_0\theta$.

Proof. Let $\xi = Q_0 \Rightarrow \ldots \Rightarrow \Box$ be successful SLD-derivation. Lemma 4.3 (ii) applied to the resultant of level $n$ of $\xi$ implies $P \models Q_0\theta_1 \ldots \theta_n$ and $Q_0\theta_1 \ldots \theta_n = Q_0(\theta_1 \ldots \theta_n |_{\text{Var}(Q_0)}) = Q_0\theta$. 

Foundations of Logic Programming

Declarative Interpretation
Comparison to Intuitive Meaning of Queries

Corollary 4.5

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$, then $P \models \exists Q_0$.

Proof.
Theorem 4.4 implies $P \models Q_0\theta$ for some $\text{CAS} \ \theta$.

Then, $P \models Q_0\theta$

implies for every interpretation $I$: if $I \models P$, then $I \models Q_0\theta$

implies for every interpretation $I$: if $I \models P$, then $I \models \forall (Q_0\theta)$

implies for every interpretation $I$: if $I \models P$, then $I \models \exists Q_0$

implies $P \models \exists Q_0$
Towards Completeness of SLD-Resolution

To show completeness of SLD-resolution we need to syntactically characterize the set of semantically derivable queries.

The concepts of term models and implication trees serve this purpose.
Term Models

$V$ set of variables, $F$ function symbols, $\Pi$ predicate symbols:

The term algebra $J$ for $F$ is defined as follows:
1. domain $D = TU_{F,V}$
2. mapping $f_J : (TU_{F,V})^n \to TU_{F,V}$ assigned to every $f \in F^n$ with
   $$f_J(t_1, \ldots, t_n) \leftrightarrow f(t_1, \ldots, t_n)$$

A term interpretation $I$ for $F$ and $\Pi$ consists of:
1. term algebra for $F$
2. $I \subseteq TB_{\Pi,F,V}$ (set of atoms that are true; equivalent: assignment of a relation $p_I \subseteq (TU_{F,V})^n$ to every $p \in \Pi^n$)

$I$ term model of a set $S$ of expressions $\iff I$ term interpretation and model of $S$
Herbrand Models

The Herbrand algebra $J$ for $F$ is defined as follows:
1. domain $D = HU_F$
2. mapping $f_j : (HU_F)^n \rightarrow HU_F$ assigned to every $f \in F^{(n)}$ with $f_j(t_1, ..., t_n) \equiv f(t_1, ..., t_n)$

A Herbrand interpretation $I$ for $F$ and $\Pi$ consists of:
1. Herbrand algebra for $F$
2. $I \subseteq HB_{\Pi,F}$ (set of ground atoms that are true)

$I$ Herbrand model of a set $S$ of expressions $:\iff$ $I$ Herbrand interpretation and model of $S$

$I$ least Herbrand model of a set $S$ of expressions $:\iff$ $I$ Herbrand model of $S$ and $I \subseteq I'$ for all Herbrand models $I'$ of $S$
Implication Trees

implication tree w.r.t. program $P$

: $\iff$

- finite tree whose nodes are atoms
- if $A$ is a node with the direct descendants $B_1, ..., B_n$ then $A \leftarrow B_1, ..., B_n \in \text{inst}(P)$
- if $A$ is a leaf, then $A \leftarrow \in \text{inst}(P)$

$E$ expression, $S$ set of expressions:

- $\text{inst}(E) \iff$ set of all instances of $E$
- $\text{inst}(S) \iff$ set of all instances of Elements $E \in S$
- $\text{ground}(E) \iff$ set of all ground instances of $E$
- $\text{ground}(S) \iff$ set of all ground instances of Elements $E \in S$
Implication Trees (Example)

Let $P_{add}$ “add-program”, $n \in \mathbb{N}$, $V$ set of variables, $t \in TU_{\{s,0\}, V}$, and

$$T = \text{add}(t, s^n(0), s^n(t))$$

$$\quad \text{add}(t, s^{n-1}(0), s^{n-1}(t))$$

$$\quad \vdots$$

$$\quad \vdots$$

$$\quad \text{add}(t, s(0), s(t))$$

$$\quad \text{add}(t, 0, t)$$

If $t \in HU_{\{s,0\}}$, then $T$ is ground implication tree w.r.t. $P_{add}$. 
Implication Trees Constitute Term Model

Lemma 4.7
Consider term interpretation $I$, atom $A$, program $P$
- $I \models A$ iff $\text{inst}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \text{inst}(P)$: if $\{B_1, \ldots, B_n\} \subseteq I$ then $A \in I$

Lemma 4.12
The term interpretation $C(P) \iff \{A \mid A \text{ is the root of some implication tree w.r.t. } P\}$ is a model of $P$. 
Ground Implication Trees Constitute Herbrand Model

Lemma 4.26
Consider Herbrand interpretation $I$, atom $A$, program $P$

- $I \models A$ iff $\text{ground}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, ..., B_n \in \text{ground}(P)$, $\{B_1, ..., B_n\} \subseteq I$ implies $A \in I$

Lemma 4.28
The Herbrand interpretation $\mathcal{M}(P) := \{A \mid A$ is the root of some ground implication tree w.r.t. $P\}$ is a model of $P$. 
Example

Let $P_{\text{add}}$ “add-program”, and $V$ set of variables.

The term interpretation
\[
C(P_{\text{add}}) = \{ \text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in TU_{\{s,0\}, V} \}
\]
\[
= \{ \text{add}(s^m(v), s^n(0), s^{n+m}(v)) \mid m, n \in \mathbb{N}, v \in V \cup \{0\} \}
\]

and the Herbrand interpretation
\[
M(P_{\text{add}}) = \{ \text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in HU_{\{s,0\}} \}
\]
\[
= \{ \text{add}(s^m(0), s^n(0), s^{n+m}(0)) \mid m, n \in \mathbb{N} \}
\]

are models of $P_{\text{add}}$. 
Correct Answer Substitutions versus Computed Answer Substitutions (Example)

Let $P_{add}$ “add-program”, and $Q = add(u, s(0), s(u))$ query.

- $\theta = \{u/s^2(v)\}$ correct answer substitution of $Q$, since $P_{add} \vdash Q\theta = add(s^2(v), s(0), s^3(v))$ (in analogy to slide 13 with $x = s^2(v)$).

- SLD-derivation of $P_{add} \cup \{Q\}$:

  $add(u, s(0), s(u)) \xrightarrow{\theta_1} add(u, 0, u) \xrightarrow{\theta_2} \square$ with $\theta_1 = \{x/u, y/0, z/u\}$ and $\theta_2 = \{x/u\}$, thus $\eta = (\theta_1 \theta_2)_{\{u\}} = \epsilon$ is a computed answer substitution of $Q$.

- Thus, $Q\eta$ more general than $Q\theta$.

- In fact, no SLD-derivation of $P_{add} \cup \{Q\}$ can deliver correct answer substitution $\theta$.  

Completeness of SLD-Resolution for Implication Trees

Query $Q$ is $n$-deep.

$$\iff$$

every atom in $Q$ is the root of an implication tree, and $n$ is the total number of nodes in these trees

Lemma 4.15

Suppose $Q\theta$ is $n$-deep for some $n \geq 0$. Then for every selection rule $R$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{CAS } \eta$ such that $Q\eta$ is more general than $Q\theta$. 
Completeness of SLD-Resolution (I)

Theorem 4.13
Suppose that $\theta$ is a correct answer substitution of $Q$. Then for every selection rule $R$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{CAS} \eta$ such that $Q\eta$ is more general than $Q\theta$.

Proof. Let $Q = A_1, ..., A_m$. Then: $\theta$ correct answer substitution of $A_1, ..., A_m$ implies $P \models A_1\theta, ..., A_m\theta$ implies for every interpretation $I$: if $I \models P$, then $I \models A_1\theta, ..., A_m\theta$ implies $C(P) \models A_1\theta, ..., A_m\theta$ (since $C(P) \models P$ by Lemma 4.12) implies $\text{inst}(A_i\theta) \subseteq C(P)$ for every $i = 1, ..., m$ (by Lemma 4.7) implies $A_i\theta \in C(P)$ for every $i = 1, ..., m$ implies $A_1\theta, ..., A_m\theta$ is $n$-deep for some $n \geq 0$ (by def. of $C(P)$) implies claim (by Lemma 4.15)
Completeness of SLD-Resolution (II)

Corollary 4.16
Suppose $P \models \exists Q$.
Then there exists a successful SLD-derivation of $P \cup \{Q\}$.

Proof. $P \models \exists Q$
implies $P \models Q\theta$ for some substitution $\theta$
implies $\theta$ correct answer substitution of $Q$
implies claim (by Theorem 4.13)
Theorem 4.29 \( \mathcal{M}(P) \) is the least Herbrand model of \( P \).

Proof. Let \( I \) be a Herbrand model of \( P \) and let \( A \in \mathcal{M}(P) \).

We prove \( A \in I \) by induction on the number \( i \) of nodes in the ground implication tree w.r.t. \( P \) with root \( A \). Then \( \mathcal{M}(P) \subseteq I \).

\( i = 1: \) A leaf implies \( A \leftarrow \in \text{ground}(P) \)
implies \( I \models A \) (since \( I \models P \))
implies \( A \in I \)

\( i \rightarrow i+1: \) A has direct descendants \( B_1, ..., B_n \) (roots of subtrees)
implies \( A \leftarrow B_1, ..., B_n \in \text{ground}(P) \) and \( B_1, ..., B_n \in I \) (induction hypothesis)
implies \( A \leftarrow B_1, ..., B_n \in \text{ground}(P) \) and \( I \models B_1, ..., B_n \)
implies \( I \models A \) (since \( I \models P \))
implies \( A \in I \)
Ground Equivalence

Theorem 4.30  For every ground atom $A$: $P \models A$ iff $\mathcal{M}(P) \models A$.

Proof. “$\Rightarrow$”: $P \models A$ and $\mathcal{M}(P) \models P$ implies $\mathcal{M}(P) \models A$ (semantic consequence).

“$\Leftarrow$”: Show for every interpretation $I$: $I \vDash P$ implies $I \vDash A$.

Let $I_H = \{A | A \text{ ground atom and } I \vDash A\}$ Herbrand interpretation.

$I \vDash P$

implies $I \vDash A \leftarrow B_1, ..., B_n$ for all $A \leftarrow B_1, ..., B_n \in \text{ground}(P)$

implies if $I \vDash B_1, ..., I \vDash B_n$ then $I \vDash A$ for all ...

implies if $B_1 \in I_H, ..., B_n \in I_H$ then $A \in I_H$ for all ... (Def. $I_H$)

implies $I_H \vDash P$ (by Lemma 4.26; thus $I_H$ Herbrand model)

implies $A \in I_H$ (since $A \in \mathcal{M}(P)$ and $\mathcal{M}(P)$ least Herbrand model)

implies $I \vDash A$ (by Def. $I_H$)
Complete Partial Orderings

Let \((\mathcal{A}, \sqsubseteq)\) be a partial ordering (cf. Slide 18 for Chapter 2).

- a least element of \(X \subseteq \mathcal{A}\)
  \(\iff a \in X, a \sqsubseteq x\) for all \(x \in X\)

- a least upper bound of \(X \subseteq \mathcal{A}\) (Notation: \(a = \sqcup X\))
  \(\iff a \in \mathcal{A}, x \sqsubseteq a\) for all \(x \in X\) and \(a\) is the least element of \(\mathcal{A}\) with this property

\((\mathcal{A}, \sqsubseteq)\) complete partial ordering (CPO) \(\iff\)

- \(\mathcal{A}\) contains a least element (denoted by \(\emptyset\))

- for every increasing sequence \(a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots\) of elements of \(\mathcal{A}\),
  the set \(X = \{a_0, a_1, a_2, \ldots\}\) has a least upper bound
Some Properties of Operators

Let \((\mathcal{A}, \sqsubseteq)\) be a CPO.

operator \(T : \mathcal{A} \rightarrow \mathcal{A}\) monotonic

\[ \iff I \sqsubseteq J \text{ implies } T(I) \sqsubseteq T(J) \]

operator \(T : \mathcal{A} \rightarrow \mathcal{A}\) finitary

\[ \iff \text{for every infinite sequence } I_0 \sqsubseteq I_1 \sqsubseteq \ldots \]

\[ \bigcup_{n=0}^{\infty} T(I_n) \text{ exists and } T\left(\bigcup_{n=0}^{\infty} I_n\right) \sqsubseteq \bigcup_{n=0}^{\infty} T(I_n) \]

operator \(T : \mathcal{A} \rightarrow \mathcal{A}\) continuous \(\iff\) \(T\) monotonic and finitary

\(I\) pre-fixpoint of \(T \iff T(I) \sqsubseteq I\)

\(I\) fixpoint of \(T \iff T(I) = I\)
Iterating Operators

Let \((A, \sqsubseteq)\) be a \text{CPO}, \(T: A \rightarrow A\), and \(I \in A\).

- \(T^{\uparrow 0} (I) :\Leftrightarrow I\)
- \(T^{\uparrow (n + 1)} (I) :\Leftrightarrow T(T^{\uparrow n} (I))\)
- \(T^{\uparrow w} (I) :\Leftrightarrow \bigsqcup_{n = 1}^{\infty} T^{\uparrow n} (I)\)

\(T^{\uparrow a} :\Leftrightarrow T^{\uparrow a} (\emptyset)\) (for \(a = 0, 1, 2, \ldots, w\))

By the definition of a \text{CPO}:
If the sequence \(T^{\uparrow 0} (I), T^{\uparrow 1} (I), T^{\uparrow 2} (I), \ldots\) is increasing, then \(T^{\uparrow w} (I)\) exists.

**Theorem 4.22**

If \(T\) is a continuous operator on a \text{CPO}, then \(T^{\uparrow w}\) exists and is the least prefixpoint of \(T\) and the least fixpoint of \(T\).
Consequence Operator

Consider the \( \text{CPO} \) (\( \{ l \mid l \text{ Herbrand interpretation} \}, \subseteq \)).
Let \( P \) be a program and \( l \) a Herbrand interpretation. Then
\[
T_P(l) :\iff \{ A \mid A \leftarrow B_1, \ldots, B_n \in \text{ground}(P), \{ B_1, \ldots, B_n \} \subseteq l \}
\]

Lemma 4.33

(i) \( T_P \) is finitary.
(ii) \( T_P \) is monotonic.
Lemma 4.32

A Herbrand interpretation $I$ is a model of $P$ iff

$$T_P(I) \subseteq I$$

Proof.

$I \models P$

iff for every $A \leftarrow B_1, ..., B_n \in \text{ground}(P)$:

$$\{B_1, ..., B_n\} \subseteq I \text{ implies } A \in I \quad \text{(by Lemma 4.26)}$$

iff for every ground atom $A$: $A \in T_P(I)$ implies $A \in I$

iff $T_P(I) \subseteq I$
Characterization Theorem

Theorem 4.34

\[ M(P) \] (i)

= least Herbrand model of \( P \) (ii)

= least pre-fixpoint of \( T_P \) (iii)

= least fixpoint of \( T_P \) (iv)

= \( T_P^{\uparrow w} \) (v)

= \{ A \mid A \text{ ground atom, } P \models A \} (vi)
success set of a program $P :\Leftrightarrow \{ A \mid A \text{ ground atom, } \exists \text{ successful SLD-derivation of } P \cup \{ A \} \}$

Theorem 4.37

For a ground atom $A$, the following are equivalent:

(i) $\mathcal{M}(P) \models A$
(ii) $P \models A$
(iii) Every SLD-tree for $P \cup \{ A \}$ is successful
(iv) $A$ is in the success set of $P$
Objectives

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models