Determinization of Weighted Tree Automata using Factorizations

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We present a determinization construction for weighted tree automata using factorizations. Among others, this result subsumes a previous result for determinization of weighted string automata using factorizations (Kirsten and Mäurer, 2005) and two previous results for weighted tree automata, one of them not using factorizations (Borchardt, 2004) and one of them restricted to nonrecursive automata over the nonnegative reals (May and Knight, 2006).

1 Introduction

Trees have applications in syntax-based approaches to both programming language semantics and natural language processing (NLP), as well as in semi-structured (e.g., XML based) databases, to name a few. In each of these applications, it is important to check whether a tree has a property; a property over trees can be viewed as a tree language, containing exactly those trees which have the property. Such a tree language is usually infinite, and a finite representation is needed to deal with it algorithmically. One formalism for a finite representation is given by tree automata [GS84, GS97], a generalization of finite-state (string) automata. In this formalism, a tree is considered in the language if its nodes can be decorated each with one state in accordance with the transitions of the automaton. Such a decoration is called a run. As in the string case, tree automata can have a nondeterministic

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behavior, admitting several runs for a single tree. Automata theoretic results like minimization or determinization are of interest because they can lead to optimal representations in terms of memory usage or processing time.

In some applications it is desirable to associate with every tree in a language some weight, e.g., to express the degree of certainty as to whether the tree is in the language or not, or to establish a preference among the trees. One example of this can be found in syntax-based NLP, where uncertainty arises from the ambiguity inherent in natural languages. A sentence (like "I saw the man with the telescope") can have several meanings, which are represented in the set of its parse trees. Weights are used to express a preference among these meanings.

Formally, a tree language with weights is modeled as a mapping from the set of all trees into some semiring A; such a mapping is called a tree series. A semiring is an algebraic structure having two binary operations, called addition and multiplication, that fulfill some algebraic laws [HW98, Gol99]. Weighted tree automata (wta) [AB87, BR82, ÉK03, FV09] are the weighted counterpart of tree automata, providing finite representations of tree series. As for tree automata, runs on a tree are considered, and a weight is assigned to each run by taking the product (using the semiring multiplication) of the transition weights. The weight of a tree is then the sum (using the semiring addition) of the weights of its runs. Again, results concerning minimization and determinization are desirable. Here we focus on determinization and show two applications.

(1) In machine translation systems, the tree series of possible translations generated from an input tree is often represented in a compact form which is, in essence, a wta [KM09]. Frequently, the wta cannot be used for further processing, and the tree series must be approximated by a list of explicitly represented trees with weight. In many cases a linear order on A is given and a good approximation is obtained by choosing "best" (i.e., highest-scoring) trees. An efficient algorithm [HC05] is known and widely used which computes a list of best runs. Clearly, this list need not be a list of best trees because a single tree may have several runs with nonzero weight—unless the automaton is deterministic. Hence, combining determinization with this algorithm yields the desired list. No efficient algorithm is known to us which does this directly.

(2) As mentioned in [Mal09], the results of [HO06, Sei90], when combined, essentially show that there is a polynomial-time minimization procedure for wta over *fields*. Apart from this special case, an efficient minimization procedure is currently only known to us for deterministic wta [Mal09].

We give a brief overview of determinization results for weighted string and tree automata. Starting from the well-known powerset construction for unweighted finite-state automata, Mohri [Moh97, Moh09] developed an algorithm for determinizing weighted finite-state automata over the tropical semiring using a factorization-based approach. His algorithm does not terminate for every given automaton, and Mohri gave a sufficient condition for termination, viz., the automata should have the twins property. By making the notion of a factorization explicit, Kirsten and Mäurer [KM05] were able to generalize this approach to arbitrary semirings. In addition to Mohri's termination condition, they also require that the factorization be maximal in some sense and that the semiring be extremal. Determinization of wta was first described by Borchardt and Vogler [BV03]. They used a Myhill-Nerode approach, which is restricted to semifields, and they showed that their construction yields a finite automaton if the semifield is locally finite. Borchardt [Bor04] extended this result to locally finite semirings using a construction more akin to the factorization approach. May and Knight [MK06] applied Mohri's approach to nonrecursive wta over the semiring of nonnegative reals, and they provided empirical evidence that their algorithm was effective in

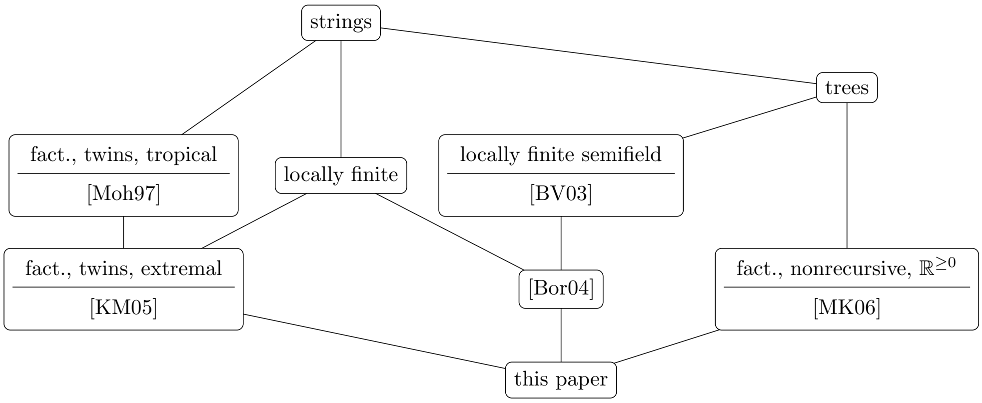


Figure 1: Concept lattice showing the relationship of different determinization results, where fact. stands for factorization.

machine translation and parsing systems, but gave no formal proof of correctness.

In this paper, we use the factorization approach to develop a determinization construction for wta which subsumes the results of [Moh97, KM05, BV03, Bor04, MK06]. This is illustrated in Fig. 1 by means of a concept lattice [Wil05]. Roughly speaking, the diagram shows which paper admits which scenarios for determinization, where a scenario consists of the input objects (strings/trees), whether a factorization is used or not, and restrictions concerning the automaton (such as nonrecursive or the twins property) and the semiring (such as extremal or locally finite). Each node represents a set of scenarios and a set of papers such that the papers admit exactly the scenarios and the scenarios are admitted by exactly the papers, and both sets are maximal in that sense. The edges read as follows: the scenarios further down include the scenarios further up, and vice-versa for the papers. Hence, the label of each node only indicates those scenarios (and papers) which are not found further up (down, respectively). For reasons of clarity, we have factored out the input objects of the scenarios.

Our paper contains the following steps. In Sect. 2, we recall some preliminaries. Section 3 contains a short introduction to weighted tree automata. In Sect. 4 the determinization construction is shown. Given a wta M and factorization (f, g), this construction yields an object $det_{(f,g)}(M)$, which differs from a deterministic wta only in that it may have an infinite number of states. Section 5 contains the results, which are (i) M and $det_{(f,g)}(M)$ are equivalent if the state set of $det_{(f,g)}(M)$ is finite (Sect. 5.1), (ii) if (f, g) is maximal in a certain sense, then the wta $det_{(f,g)}(M)$ has at most as many states as $det_{(f',g')}(M)$ for an arbitrary factorization (f', g') (Sect. 5.2), (iii) sufficient conditions for $det_{(f,g)}(M)$ to have a finite number of states, most prominently this is the case if the semiring is extremal and M has the twins property (Sect. 5.3), (iv) the twins property is decidable for cycle-unambiguous wta (Sect. 5.4). Results (i)–(iii) are summarized in our main theorem (Theorem 5.2), which also substantiates the position of this paper in Fig. 1. Starting with Sect. 5, the proofs in this paper are organized in a top-down fashion, i.e., most lemmata are employed before they are proved.

Our determinization result was largely obtained by generalizing [KM05] from strings to trees, and most of our proofs follow theirs. The decidability result is adapted from [AM03], again from strings to trees; however, our proof does not follow theirs. We note that this transition from strings to trees does incur considerable combinatorial blowup.

We also note that the key ideas of this paper have been presented at FSMNLP 2009 [BMV09], but we did not submit our work to the proceedings.

2 Preliminaries

A *ranked alphabet* is a pair (Σ, rk) where Σ is an alphabet, i.e., a finite set, and $rk : \Sigma \to \mathbb{N}$. If rk is clear from the context, then for every $k \in \mathbb{N}$ we write $\Sigma^{(k)}$ instead of $rk^{-1}(k)$. Henceforth, we identify (Σ, rk) with Σ.

Let V be a set. We define the *set of trees over* Σ *indexed by* V, denoted by $T_\Sigma(V)$, to be the smallest set T such that $V \subseteq T$ and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \dots, \xi_k \in T$, also $\sigma(\xi_1, \dots, \xi_k) \in T$. We set $T_\Sigma = T_\Sigma(\emptyset)$. Let $\xi \in T_\Sigma(V)$. For every $\xi \in T_\Sigma(V)$, we define the set $pos(\xi) \subseteq \mathbb{N}^*$ of *positions of* ξ as follows. If $\xi \in V$, then $pos(\xi) = \{\varepsilon\}$. If $\xi = \sigma(\xi_1, \dots, \xi_k)$, then $pos(\xi) = \{\varepsilon\} \cup \{iw \mid i \in \{1, \dots, k\}, w \in pos(\xi_i)\}$. We will use the strict and the nonstrict prefix order on $pos(\xi)$, denoted by $<$ and $\leq$, respectively. Let $w \in pos(\xi)$. The label of ξ at w is denoted by $\xi(w)$ and the subtree of ξ rooted at w is denoted by $\xi|_w$.

A *semiring* [HW98, Gol99] is a tuple $\mathcal{A} = (A, +, \cdot, 0, 1)$ where A is a set, $+$ and $\cdot$ are binary operations on A, and 0 and 1 are elements of A such that:

- $(A, +, 0)$ is a commutative monoid, i.e.,
  - $+$ is *associative* $(a + (b + c) = (a + b) + c$ for every $a, b, c \in A)$,
  - $+$ is *commutative* $(a + b = b + a$ for every $a, b \in A)$, and
  - 0 is *neutral with respect to* $+$ $(a + 0 = a$ for every $a \in A)$,
- $(A, \cdot, 1)$ is a monoid, i.e., $\cdot$ is associative and 1 is neutral with respect to $\cdot$,
- $\cdot$ distributes over $+$ $(a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for every $a, b, c \in A)$,
- 0 is *absorbing with respect to* $\cdot$ $(a \cdot 0 = 0 = 0 \cdot a$ for every $a \in A)$.

Let $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring. In notation, we will identify $\mathcal{A}$ with A. We say that A is *commutative* if the operation $\cdot$ is commutative, A is *zero-divisor free* if $a \cdot b = 0$ implies that $a = 0$ or $b = 0$, A is *zero-sum free* if $a + b = 0$ implies that $a = b = 0$, and A is a *semifield* if it admits multiplicative inverses, i.e., for every $a \in A \setminus \{0\}$ there is a uniquely determined $a^{-1} \in A$ such that $a \cdot a^{-1} = 1$. Moreover, A is *locally finite* if for every finite subset $A' \subseteq A$ the closure of A' under 0, 1, $+$, and $\cdot$ is finite.

Example 2.1 We give examples of semirings (commutative except no. 5):

1. the *Boolean semiring* $(\mathbb{B}, \vee, \wedge, 0, 1)$ where $\mathbb{B} = \{0, 1\}$, and $\vee$ and $\wedge$ denote disjunction and conjunction, respectively;
2. the *semiring of nonnegative real numbers* $(\mathbb{R}^{\geq 0}, +, \cdot, 0, 1)$;
3. the *Viterbi semiring* $([0, 1], \max, \cdot, 0, 1)$;
4. the *tropical semiring* $(\mathbb{R}^{\geq 0}_\infty, \min, +, \infty, 0)$ where $\mathbb{R}^{\geq 0}_\infty = \mathbb{R}^{\geq 0} \cup \{\infty\}$ and min and $+$ are extended to $\mathbb{R}^{\geq 0}_\infty$ in the obvious way;

5. the *formal language semiring* $(\mathcal{P}(\Delta^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ over an alphabet Δ where $\cdot$ denotes the usual concatenation of languages. □

For the rest of this paper, let Σ *be an arbitrary ranked alphabet, and let* $(A, +, \cdot, 0, 1)$ *be an arbitrary semiring.*

Let Q be a set. The set of all mappings from Q into A (in some cases also called *Q-vectors over A*) is denoted by A^Q. For every $u \in A^Q$ and $q \in Q$, we denote $u(q)$ by u_q, the *q-component of u*. The Q-vector over A which maps every $q \in Q$ to 0 is denoted by $\tilde{0}$.

A *(formal) tree series (over* Σ *and* A*)* is a mapping $\varphi : T_\Sigma \to A$. For every $\xi \in T_\Sigma$, we call $\varphi(\xi)$ the *coefficient of* ξ *with respect to* φ and denote it by (φ, ξ). The set of all tree series over Σ and A is denoted by $A\langle\langle T_\Sigma\rangle\rangle$. The tree series which maps every tree to 0 is denoted by $\tilde{0}$.

3 Weighted Tree Automata

A wta [AB87, BR82, ÉK03, FV09] is a finite-state device which specifies a tree series. The core of a wta is its transition mapping. Imagine a situation where the automaton processes some node w of some input tree ξ. Then a transition of the automaton is described by the states at the direct descendants of w, the label at w, and the state at w itself. The transition mapping assigns a weight to every possible transition. The transition weights are used later on to define the weight of ξ. Intuitively, assigning a weight of 0 means that a transition is not possible.

A *wta (over* Σ *and* A*)* is a triple $M = (Q, \mu, \nu)$ where Q is a nonempty finite set (of *states*), μ is a family $(\mu_k \mid k \in \mathbb{N})$ (of *transition mappings*), $\mu_k : \Sigma^{(k)} \to A^{Q^k \times Q}$ for every $k \in \mathbb{N}$, and $\nu \in A^Q$ (*final weights*). Let $M = (Q, \mu, \nu)$ be a wta. We say that M is *bottom-up deterministic* (for short: *bu-det*) if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(q_1, \ldots, q_k) \in Q^k$, there is at most one $q \in Q$ such that $\mu_k(\sigma)_{q_1 \ldots q_k, q} \neq 0$.[1] Trivially, if $|Q| = 1$, then M is bu-det.

Example 3.1 (Running example) Let A be the Viterbi semiring $([0,1], \max, \cdot, 0, 1)$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, and $M = (Q, \mu, \nu)$ the wta over Σ and A where

- $Q = \{Z, B\}$,
- $\mu_0(\alpha)_{\varepsilon,B} = 1$, $\mu_0(\alpha)_{\varepsilon,Z} = 0.2$, $\mu_2(\sigma)_{BZ,Z} = 0.5$, and $\mu_2(\sigma)_{w,q} = 0$ for every $(w, q) \in (Q^2 \times Q) \setminus \{(BZ, Z)\}$,
- $\nu_Z = 1$, $\nu_B = 0$.

The family μ can be visualized by a hypergraph as shown in Fig. 2, which omits any transitions with weight 0. Another equivalent representation of μ is given by the following rules and rule weights of a *weighted regular tree grammar* [AB87]:

$$\begin{array}{ll} Z \to \sigma(B, Z) & 0.5 \\ Z \to \alpha & 0.2 \\ B \to \alpha & 1 \end{array}$$

Note that M is not bu-det because we have both $\mu_0(\alpha)_{\varepsilon,B} \neq 0$ and $\mu_0(\alpha)_{\varepsilon,Z} \neq 0$. □

[1] We identify the k-tuple $(q_1, \ldots, q_k)$ with the string $q_1 \ldots q_k$.

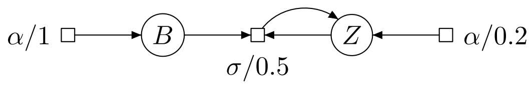


Figure 2: Visualization of a transition mapping using a hypergraph.

Now we recall from [FV09] two different ways of giving semantics to wta. While the resulting semantics are equal, each way offers its unique perspective, which will facilitate our considerations. We begin with the *initial algebra semantics*, which provides the framework for both our determinization construction and the proof of its correctness. In this approach, in order to compute the weight for a given input tree $\xi = \sigma(\xi_1, \dots, \xi_k)$, the wta recursively computes for each state a separate weight of ξ. Let us denote the weight of ξ in state q by $h_\mu(\xi)_q$. Then $h_\mu(\xi)$ is a Q-vector over A, i.e., $h_\mu(\xi) \in A^Q$. The vector $h_\mu(\xi)$ is obtained by combining the weights from the vectors $h_\mu(\xi_1), \dots, h_\mu(\xi_k)$ in a specific way, using the transition weights from μ. For this we define an auxiliary mapping which is an extension of μ.

Let $M = (Q, \mu, \nu)$. Then μ induces a family $(\mu_M(\sigma) \mid \sigma \in \Sigma)$ of mappings where for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$ we have $\mu_M(\sigma) : \underbrace{A^Q \times \dots \times A^Q}_{k} \to A^Q$ and for every $u_1, \dots, u_k \in A^Q$

$$\mu_M(\sigma)(u_1, \dots, u_k)_q = \sum_{(q_1,\dots,q_k) \in Q^k} (u_1)_{q_1} \cdot \ldots \cdot (u_k)_{q_k} \cdot \mu_k(\sigma)_{q_1 \dots q_k, q} \; .$$

Then $h_\mu : T_\Sigma \to A^Q$ is defined recursively by letting $h_\mu(\xi) = \mu_M(\sigma)(h_\mu(\xi_1), \dots, h_\mu(\xi_k))$ for every $\xi = \sigma(\xi_1, \dots, \xi_k)$. Finally, the *i-behavior of* M, denoted by $[\![M]\!]_i$, is the tree series in $A\langle\langle T_\Sigma \rangle\rangle$ such that for every $\xi \in T_\Sigma$:

$$([\![M]\!]_i, \xi) = \sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q \; .$$

In fact, h_μ is the unique Σ-homomorphism from the initial Σ-algebra T_Σ to the Σ-algebra (A^Q, μ_M); this gives the name to this semantics [GTWW77].

Example 3.2 (Example 3.1 contd.) We show $[\![M]\!]_i$. For notational convenience, we will write the elements of A^Q as column vectors, where the first row is the B-component. By elementary computation, we have

$$h_\mu(\alpha) = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} , \qquad h_\mu(\sigma(\alpha, \alpha)) = \begin{pmatrix} 0 \\ h_\mu(\alpha)_B \cdot h_\mu(\alpha)_Z \cdot 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} .$$

Now we form a general hypothesis. To this end, we define for every $n \in \mathbb{N}$ the tree $\rho_n \in T_\Sigma$ by letting $\rho_0 = \alpha$ and $\rho_{n+1} = \sigma(\alpha, \rho_n)$ for every $n \in \mathbb{N}$. It is easy to prove by induction on ξ that

$$h_\mu(\xi)_Z = \begin{cases} 0.2 \cdot 0.5^n & \text{if } \xi = \rho_n \text{ for some } n \in \mathbb{N} \; , \\ 0 & \text{otherwise} \; . \end{cases}$$

By the nature of ν, it is easy to see that $([\![M]\!]_i, \xi) = h_\mu(\xi)_Z$. □

In the *run semantics*, which provides the framework for our finiteness result, every node of the input tree is decorated with a state. Using the information from μ, such a decoration

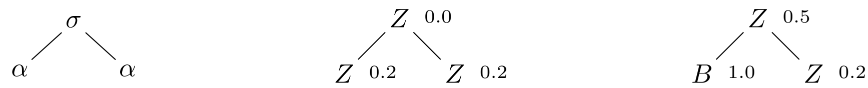


Figure 3: Visualization of an input tree ξ and runs κ_1 and κ_2 from Example 3.3.

(called a *run*) is given a weight. The weight of the input tree is then the sum of the weights of all possible runs. Preparing for our finiteness result, we consider trees with zero or more occurrences of the nullary symbol z.

Let $\xi \in T_\Sigma(\{z\})$ and $p, q \in Q$. We define the set $R_M^{p;q}(\xi)$ of *runs of M on ξ beginning at occurrences of z in state p and ending at the root in state q* by letting $R_M^{p;q}(\xi) = \{\kappa : pos(\xi) \to Q \mid \kappa(\varepsilon) = q, \forall w \in pos(\xi)\colon \xi(w) = z \text{ implies } \kappa(w) = p\}$. We set $R_M^q(\xi) = \bigcup_{p \in Q} R_M^{p;q}(\xi)$ and $R_M(\xi) = \bigcup_{q \in Q} R_M^q(\xi)$.

Moreover, we define the mapping $wt_{M,\xi} : R_M(\xi) \to A$ by letting for every $\kappa \in R_M(\xi)$: $wt_{M,\xi}(\kappa) = \prod_{w \in pos(\xi)} wt_{M,\xi}(\kappa, w)$ where multiplication is done in postorder traversal, $wt_{M,\xi}(\kappa, w) = 1$ if $\xi(w) = z$, and $wt_{M,\xi}(\kappa, w) = \mu_k(\sigma)_{\kappa(w1)\dots\kappa(wk),\kappa(w)}$ if $\xi|_w = \sigma(\xi_1, \dots, \xi_k)$. Furthermore, we will use the following abbreviation: for every finite subset $R \subseteq R_M(\xi)$ we let $wt_{M,\xi}(R) = \sum_{\kappa \in R} wt_{M,\xi}(\kappa)$.

The *r-behavior of M*, denoted by $[\![M]\!]_r$, is the tree series in $A\langle\!\langle T_\Sigma \rangle\!\rangle$ such that for every $\xi \in T_\Sigma$:

$$([\![M]\!]_r, \xi) = \sum_{q \in Q} wt_{M,\xi}(R_M^q(\xi)) \cdot \nu_q \ .$$

Example 3.3 (Example 3.1 contd.) Fig. 3 depicts the tree $\xi = \sigma(\alpha, \alpha)$ and two runs $\kappa_1, \kappa_2 \in R_M^Z(\xi)$, augmented by the transition weights at each position. It is easy to see that $wt_{M,\xi}(\kappa_1) = 0$ and $wt_{M,\xi}(\kappa_2) = 0.1$. □

In [FV09, Sect. 3.2], it has been shown that $[\![M]\!]_i = [\![M]\!]_r$. Hence, we drop the subscript from these notations. We say that two wta M and M' over Σ and A are *equivalent* if $[\![M]\!] = [\![M']\!]$.

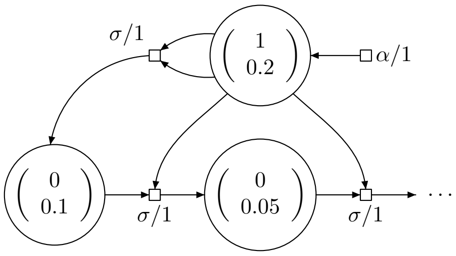
Observation 3.4 *Let (Q, μ, ν) be a bu-det wta and $\xi \in T_\Sigma$. There is at most one $q \in Q$ such that $\mathrm{h}_\mu(\xi)_q \neq 0$, and there is at most one $\kappa \in R_M(\xi)$ such that $wt_{M,\xi}(\kappa) \neq 0$.*

It is easy to see that for every wta $M = (Q, \mu, \nu)$ such that $\mu_M(\alpha)() = \tilde{0}$ for every $\alpha \in \Sigma^{(0)}$, we have $[\![M]\!] = \tilde{0}$. Since this tree series can also be computed by a wta which does not have this property, the following assumption does not imply a loss of generality.

> *In the sequel, let $M = (Q, \mu, \nu)$ be an arbitrary wta over Σ and A such that $\mu_M(\alpha)() \neq \tilde{0}$ for some $\alpha \in \Sigma^{(0)}$. Also, if confusion is ruled out, we will omit the subscripts M and ξ from μ_M, $wt_{M,\xi}$, and R_M.*

4 Factorizations and Determinization

In the unweighted case, determinization is accomplished using the well-known powerset construction. As shown in [Bor04], a straightforward generalization of this construction to the weighted case is obtained by using the state behavior of the new automaton to simulate the semantic algebra (A^Q, μ). If A is not locally finite, this may yield an infinite set of states.



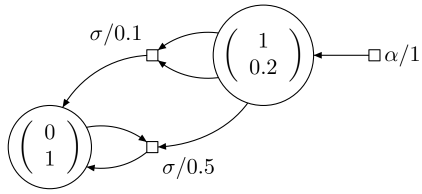


Figure 4: Determinization of the wta M from Example 3.1 via (a) the approach of Borchardt [Bor04] and (b) the factorization approach.

Example 4.1 (Example 3.2 contd.) We apply the technique of [Bor04] to M. The new state set is obtained as follows:

$$Q' = \{h_\mu(\xi) \mid \xi \in T_\Sigma\} = \{h_\mu(\rho_i) \mid i \in \mathbb{N}\} \cup \{h_\mu(\xi) \mid \xi \in T_\Sigma \setminus \{\rho_i \mid i \in \mathbb{N}\}\}$$
$$= \{\begin{pmatrix} 1 \\ 0.2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.05 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.025 \end{pmatrix}, \ldots\} \cup \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}\,.$$

Part of the resulting transition mapping of the construction is shown in Fig. 4(a). Notice how the determinized automaton mimics the behavior of μ_M using its states. □

By adapting the factorization technique of [KM05], we are able to improve on the previous approach. Instead of moving the complete computation of weights from the semantic algebra (A^Q, μ) into the new states, we factor the elements of A^Q in such a way that the transition mapping in the new automaton is equipped with the factor common to all components. For instance, applying this technique to the wta of Fig. 2 yields the wta in Fig. 4(b). Now we develop this approach in more detail.

Let Q be a nonempty finite set. A pair (f, g) is a *factorization (of dimension Q)* if $f : A^Q \setminus \{\tilde{0}\} \to A^Q$, $g : A^Q \setminus \{\tilde{0}\} \to A$, and $u = g(u) \cdot f(u)$ for every $u \in A^Q \setminus \{\tilde{0}\}$. A factorization (f, g) is called *maximal* if for every $u \in A^Q$ and $a \in A$, we have that $a \cdot u \neq \tilde{0}$ implies $f(u) = f(a \cdot u)$. Note that even if $f(\tilde{0})$ were defined, the case $a \cdot u = \tilde{0}$ would still have to be excluded here, because otherwise we would obtain that $f(u) = f(0 \cdot u) = f(\tilde{0})$ for every $u \in A^Q$. Also note that, for every finite set Q, we have the uniquely defined *trivial* factorization (f, g) where $f(u) = u$ and $g(u) = 1$ for every $u \in A^Q \setminus \{\tilde{0}\}$. When the trivial factorization is used, our determinization construction equals that of [Bor04].

The following lemma shows a maximal factorization for zero-sum-free semifields.

Lemma 4.2 *Let Q be a nonempty finite set and let (f, g) is a maximal factorization where for every $u \in A^Q \setminus \{\tilde{0}\}$ we let $g(u) = \sum_{q \in Q} u_q$, and $f(u) = g(u)^{-1} \cdot u$.*

Proof. We show that (f, g) is a factorization. Let $u \in A^Q \setminus \{\tilde{0}\}$. Since A is zero-sum free, $g(u) \neq 0$ and hence, $g(u) \cdot f(u) = g(u) \cdot g(u)^{-1} \cdot u = u$. We show that (f, g) is maximal.

Let $a \in A$ such that $a \cdot u \neq \tilde{0}$. Moreover, let $q \in Q$. Then

$$\begin{aligned}[f(a \cdot u)]_q &= \left[g(a \cdot u)^{-1} \cdot a \cdot u\right]_q = \left(\textstyle\sum_{q' \in Q} a \cdot u_{q'}\right)^{-1} \cdot a \cdot u_q \\ &= \left(a \cdot \textstyle\sum_{q' \in Q} u_{q'}\right)^{-1} \cdot a \cdot u_q = \left(\textstyle\sum_{q' \in Q} u_{q'}\right)^{-1} \cdot a^{-1} \cdot a \cdot u_q \\ &= g(u)^{-1} \cdot u_q = [f(u)]_q \ .\end{aligned}$$

■

Note that we could have added an arbitrary nonzero constant factor in our definition of g. This shows that the maximal factorization need not be unique.

Example 4.3 Let Q be a nonempty finite set and (f, g) a factorization of dimension Q. We show some cases concerning A and (f, g) where (f, g) is maximal:

1. if A is the semiring of nonnegative reals $(\mathbb{R}^{\geq 0}, +, \cdot, 0, 1)$, $g(u) = \sum_{q \in Q} u_q$, and $f(u) = \frac{1}{g(u)} \cdot u$;
2. if A is the semiring $(\mathbb{R}^{\geq 0}, \max, \cdot, 0, 1)$, $g(u) = \sum_{q \in Q} u_q$, and $f(u) = \frac{1}{g(u)} \cdot u$;
3. if A is the Viterbi semiring $([0, 1], \max, \cdot, 0, 1)$, $g(u) = \max\{u_q \mid q \in Q\}$, and $f(u) = \frac{1}{g(u)} \cdot u$;
4. if A is the tropical semiring $(\mathbb{R}^{\geq 0}_{\infty}, \min, +, \infty, 0)$, $g(u) = \min\{u_q \mid q \in Q\}$, and $f(u) = -g(u) + u$.

The constructions of [Moh97] and [MK06] (implicitly) employ the maximal factorizations given here for the tropical semiring and the semiring of nonnegative reals, respectively. □

The following lemma shows that (apart from the case that $|Q| = 1$) maximal factorizations only exist for zero-divisor-free semirings.

Lemma 4.4 *Let (f, g) be a maximal factorization. Then $|Q| = 1$ or A is zero-divisor free.*

Proof. We show this by contradiction. Assume that $|Q| > 1$ and A has zero divisors, i.e., $a_1, a_2 \in A \setminus \{0\}$ such that $a_1 \cdot a_2 = 0$. We choose a pair $q_1, q_2 \in Q$ such that $q_1 \neq q_2$. This is possible because $|Q| > 1$. We define the vectors $u_i \in A^Q$ as the vectors whose q_i-component is 1 while the other components are 0. Let $U_1 = \{a \cdot u_1 \mid a \in A \setminus \{0\}\}$ and $U_2 = \{u_1 + a_2 \cdot u_2\}$. We claim that (i) $U_1 \cup U_2 \subseteq f^{-1}(f(u_1))$, (ii) $U_1 \cap U_2 = \emptyset$, and (iii) $g(f^{-1}(f(u_1))) \subseteq A \setminus \{0\}$, from which we can easily derive

$$|f^{-1}(f(u_1))| \geq |U_1 \cup U_2| = |A| > |A \setminus \{0\}| \geq |g(f^{-1}(f(u_1)))|,$$

which is a contradiction to the fact that (f, g) is a factorization.

Now we show (i). Clearly, $f(a \cdot u_1) = f(u_1)$ for every $a \in A \setminus \{0\}$ because the factorization (f, g) is maximal. In addition, we obtain $f(u_1) = f(a_1 \cdot u_1) = f(a_1 \cdot u_1 + a_1 \cdot (a_2 \cdot u_2)) = f(u_1 + a_2 \cdot u_2)$ because $a_1 \cdot a_2 \cdot u_2 = \tilde{0}$.

Statement (ii) is easy to see.

For the proof of (iii), assume that there is a $u \in A^Q \setminus \{\tilde{0}\}$ such that $g(u) = 0$. Since (f, g) is a factorization, we can derive $u = g(u) \cdot f(u) = 0 \cdot f(u) = \tilde{0}$. This is a contradiction to our assumption that $u \in A^Q \setminus \{\tilde{0}\}$. Hence, $g(A^Q \setminus \{\tilde{0}\}) \subseteq A \setminus \{0\}$. ■

We now adapt the method of determinization by factorization [KM05, Sect. 3.3] to the tree case. Let (f, g) be a factorization of dimension Q. The *determinization of* M *by* (f, g) is the triple $det_{(f,g)}(M) = (Q', \mu', \nu')$ where

- Q' is the smallest set $P \subseteq A^Q$ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $u_1, \dots, u_k \in P$: if $\mu(\sigma)(u_1, \dots, u_k) \neq \tilde{0}$, then $f(\mu(\sigma)(u_1, \dots, u_k)) \in P$.
- $\mu'_k(\sigma)_{u_1 \dots u_k, u} = \begin{cases} g(\mu(\sigma)(u_1, \dots, u_k)) & \text{if } \mu(\sigma)(u_1, \dots, u_k) \neq \tilde{0} \text{ and} \\ & \quad u = f(\mu(\sigma)(u_1, \dots, u_k)) \text{ ,} \\ 0 & \text{otherwise ,} \end{cases}$

  for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $u_1, \dots, u_k \in Q'$, and $u \in Q'$,
- $\nu'_u = \sum_{q \in Q} u_q \cdot \nu_q$ for every $u \in Q'$.

Note that Q' is uniquely determined because it is chosen from a set which is closed under intersection. Moreover, Q' is not empty because of our assumption that $\mu(\alpha)() \neq \tilde{0}$ for some $\alpha \in \Sigma^{(0)}$. Hence, $det_{(f,g)}(M)$ is a wta iff Q' is finite. It is easy to see that, if $det_{(f,g)}(M)$ is a wta, then it is bu-det.

The following observation, which can be proved using the fixpoint theorem of Tarski and Kleene [Wec92, Sect. 1.5.2], shows a stratification of Q'; this basically gives an algorithm for computing Q' (in case it is finite).

Observation 4.5 *Let* $det_{(f,g)}(M) = (Q', \mu', \nu')$*. Then there is a family* $(Q'_i \mid i \in \mathbb{N})$ *such that* $Q'_0 = \emptyset$*, for every* $i \in \mathbb{N}$ *we have* $Q'_{i+1} = \{f(\mu(\sigma)(u_1, \dots, u_k)) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, u_1, \dots, u_k \in Q'_i, \mu(\sigma)(u_1, \dots, u_k) \neq \tilde{0}\}$ *and* $Q' = \bigcup_{i \in \mathbb{N}} Q'_i$*; moreover,* Q' *is finite iff there is an* $n \in \mathbb{N}$ *with* $Q' = Q'_n$*.*

Example 4.6 (Example 3.1 contd.) We show how $det_{(f,g)}(M) = (Q', \mu', \nu')$ is computed using the maximal factorization (f, g) given for the Viterbi semiring in Example 4.3. First, we compute Q' using Observation 4.5. We use the following abbreviations:

$$u_1 = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} \qquad \text{and} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Then $Q'_0 = \emptyset$ and

$$\begin{aligned} Q'_1 &= \{f(\mu(\alpha)())\} = \{f(u_1)\} = \{u_1\} \text{ ,} \\ Q'_2 &= \{f(\mu(\alpha)()), f(\mu(\sigma)(u_1, u_1))\} = Q'_1 \cup \{f(0.1 \cdot u_2)\} = \{u_1, u_2\} \text{ ,} \\ Q'_3 &= Q'_2 \cup \{f(\mu(\sigma)(u_1, u_2))\} = Q'_2 \cup \{f(0.5 \cdot u_2)\} = \{u_1, u_2\} \text{ ,} \end{aligned} \tag{$\star$}$$

where we note for $(\star)$ that $\mu(\sigma)(u_2, u_1) = \tilde{0} = \mu(\sigma)(u_2, u_2)$. Clearly, we have $Q'_i = Q'_2$ for every $i \in \mathbb{N}$, $i \geq 2$. Hence, $Q' = Q_2$. Figure 4(b) shows μ'. In particular, we can read off from the above calculation that

$$\begin{aligned} \mu'_2(\sigma)_{u_1 u_1, u_2} &= g(\mu(\sigma)(u_1, u_1)) = g(0.1 \cdot u_2) = 0.1 \text{ ,} \\ \mu'_2(\sigma)_{u_1 u_2, u_2} &= g(\mu(\sigma)(u_1, u_2)) = g(0.5 \cdot u_2) = 0.5 \text{ ,} \end{aligned}$$

and that $\mu'_2(\sigma)$ is 0 on every other index. Finally, $\nu'_{u_1} = 0.2$ and $\nu'_{u_2} = 1$. □

5 Results

This section contains the results, which are in brief (i) correctness of our construction, (ii) a minimality property, (iii) four sufficient conditions for $det_{(f,g)}(M)$ to be a wta, and (iv) decidability of the twins property for a decidable subclass of wta. Each of these items is shown in a separate subsection. We start with the definitions they require and we state our main theorem (Theorem 5.2) which summarizes results (i)–(iii).

The first sufficient condition for $det_{(f,g)}(M)$ to be a wta requires M to be nonrecursive, thus providing a formal verification of [MK06]. The second condition is adapted from [Bor04], i.e., it requires A to be locally finite. The remaining conditions share the requirement of a maximal factorization. In addition, the third one requires M to be bu-det. The fourth condition is adapted from [KM05, Theorem 5], that is, it requires A to be *extremal*[2] [Mah84] and M to have the *twins property* [Cho77].

The semiring A is called *extremal* if $a + b \in \{a, b\}$ for every $a, b \in A$. For instance, the Viterbi semiring and the tropical semiring are extremal.

To define the twins property we need the concept of a context and the concept of substitution into a context. A Σ-*context* is a tree in $T_\Sigma(\{z\})$ in which z occurs exactly once. The set of all Σ-contexts is denoted by C_Σ. Given a context ζ and a tree ξ, we denote by $\zeta \cdot \xi$ the tree which is obtained from ζ by replacing the occurrence of z by ξ.

A wta $M = (Q, \mu, \nu)$ is said to have the *twins property* if for every $p, q \in Q$, $\xi \in T_\Sigma$, and $\zeta \in C_\Sigma$, we have that if $wt(R^{p,p}(\zeta)) \neq 0$, $wt(R^p(\xi)) \neq 0$, $wt(R^{q,q}(\zeta)) \neq 0$, and $wt(R^q(\xi)) \neq 0$, then $wt(R^{p,p}(\zeta)) = wt(R^{q,q}(\zeta))$.

Example 5.1 (Example 3.1 contd.) We show that M has the twins property. Let $p, q \in Q$, $\xi \in T_\Sigma$, and $\zeta \in C_\Sigma$ such that $wt(R^{p,p}(\zeta)) \neq 0$, $wt(R^p(\xi)) \neq 0$, $wt(R^{q,q}(\zeta)) \neq 0$, and $wt(R^q(\xi)) \neq 0$. We show that $wt(R^{p,p}(\zeta)) = wt(R^{q,q}(\zeta))$. If $p = q$, this is trivial. For reasons of symmetry, it suffices to consider the case that $p = B$ and $q = Z$. Since $wt(R^{B,B}(\zeta)) \neq 0$ and $wt(R^B(\xi)) \neq 0$, we conclude that $\zeta = z$ and $\xi = \alpha$. Thus we obtain $wt(R^{B,B}(z)) = 1 = wt(R^{Z,Z}(z))$. □

We note that the twins property as given here is undecidable, i.e., there is no algorithm that for any given wta M outputs whether M has the twins property. This follows from the fact that equality of recognizable power series over the tropical semiring is undecidable [Kro94]. However, the twins property is decidable for cycle-unambiguous wta (see Sect. 5.4).

Theorem 5.2 *Let $M = (Q, \mu, \nu)$ be a wta, (f, g) be the trivial or a maximal factorization, and A commutative if (f, g) is maximal. Moreover, let one of the following conditions hold:*

- *M is nonrecursive,*
- *A is locally finite,*
- *(f, g) is maximal and M is bu-det, or*
- *(f, g) is maximal, M has the twins property, and A is extremal.*

Then $\det_{(f,g)}(M)$ is a bu-det wta and $[\![M]\!] = [\![det_{(f,g)}(M)]\!]$. Moreover, if (f, g) is maximal, then regarding the number of states, $\det_{(f,g)}(M)$ is minimal among all wta which are obtained by factorization.

[2] called *min-semiring* in [KM05]

PROOF. The statements of the theorem are a consequence of Theorems 5.3, 5.5, 5.7, and 5.8. ■

The reader is invited to compare Theorem 5.2 to the overview given in Fig. 1. Note that [KM05] also require a commutative semiring for their finiteness result.

In the subsections to come, we prove the individual results.

5.1 Correctness

In this section, we show that our determinization construction is correct; more precisely: if $det_{(f,g)}(M)$ is a wta, then $det_{(f,g)}(M)$ is equivalent to M. The following theorem corresponds to Theorem 1 of [KM05].

Theorem 5.3 *Let (f, g) be a factorization. If (f, g) is not the trivial factorization, then let A be commutative. If $\det_{(f,g)}(M)$ is a wta, then $[\![M]\!] = [\![det_{(f,g)}(M)]\!]$.*

PROOF. We abbreviate $det_{(f,g)}(M)$ by M'. Let $M' = (Q', \mu', \nu')$. Let M' be a wta, i.e., let Q' be finite. We show the following statement by induction on ξ: for every $\xi \in T_\Sigma$ we have $h_\mu(\xi) = \sum_{u \in Q'} h_{\mu'}(\xi)_u \cdot u$. Note that, if (f, g) is the trivial factorization, then $h_{\mu'}(\xi)_u \in \{0, 1\}$ for every $u \in Q'$.

Let $\xi = \sigma(\xi_1, \dots, \xi_k)$. By Obs. 3.4 there are $u'_1, \dots, u'_k \in Q'$ such that for every $i \in \{1, \dots, k\}$ and $u \in Q' \setminus \{u'_i\}$ we have $h_{\mu'}(\xi_i)_u = 0$. We derive $(\star)$:

$$\begin{aligned} h_\mu(\xi) &= \mu(\sigma)(h_\mu(\xi_1), \dots, h_\mu(\xi_k)) \\ &= \mu(\sigma)(h_{\mu'}(\xi_1)_{u'_1} \cdot u'_1, \dots, h_{\mu'}(\xi_k)_{u'_k} \cdot u'_k) && \text{(induction hypothesis)} \\ &= h_{\mu'}(\xi_1)_{u'_1} \cdot \dots \cdot h_{\mu'}(\xi_k)_{u'_k} \cdot \mu(\sigma)(u'_1, \dots, u'_k) && \text{(Obs. 5.4)} \end{aligned}$$

and $(\dagger)$: for every $u \in Q'$

$$\begin{aligned} h_{\mu'}(\xi)_u &= \mu(\sigma)(h_{\mu'}(\xi_1), \dots h_{\mu'}(\xi_k))_u \\ &= \textstyle\sum_{u_1, \dots, u_k \in Q'} h_{\mu'}(\xi_1)_{u_1} \cdot \dots \cdot h_{\mu'}(\xi_k)_{u_k} \cdot \mu'_k(\sigma)_{u_1 \dots u_k, u} \\ &= h_{\mu'}(\xi_1)_{u'_1} \cdot \dots \cdot h_{\mu'}(\xi_k)_{u'_k} \cdot \mu'_k(\sigma)_{u'_1 \dots u'_k, u} \ . \end{aligned}$$

Now we distinguish two cases.

If $\mu(\sigma)(u'_1, \dots, u'_k) = \tilde{0}$, then $(\star)$ implies $h_\mu(\xi) = \tilde{0}$. By definition, we have $\mu'_k(\sigma)_{u'_1 \dots u'_k, u} = 0$ for every $u \in Q'$. Hence, $(\dagger)$ implies $h_{\mu'}(\xi) = \tilde{0}$.

If $\mu(\sigma)(u'_1, \dots, u'_k) \neq \tilde{0}$, we set $u' = f(\mu(\sigma)(u'_1, \dots, u'_k)$ and derive

$$\begin{aligned} h_\mu(\xi) &= h_{\mu'}(\xi_1)_{u'_1} \cdot \dots \cdot h_{\mu'}(\xi_k)_{u'_k} \cdot \mu(\sigma)(u'_1, \dots, u'_k) && (\star) \\ &= (h_{\mu'}(\xi_1)_{u'_1} \cdot \dots \cdot h_{\mu'}(\xi_k)_{u'_k} \cdot g(\mu(\sigma)(u'_1, \dots, u'_k))) \cdot u' && ((f,g) \text{ fact.}) \\ &= h_{\mu'}(\xi)_{u'} \cdot u' && (\dagger) \\ &= \textstyle\sum_{u \in Q'} h_{\mu'}(\xi)_u \cdot u \ . && \text{(Obs. 3.4)} \end{aligned}$$

Now we show that $([\![M']\!], \xi) = ([\![M]\!], \xi)$.

$$\begin{aligned} ([\![M']\!], \xi) &= \textstyle\sum_{u \in Q'} h_{\mu'}(\xi)_u \cdot \nu'_u \\ &= \textstyle\sum_{u \in Q'} h_{\mu'}(\xi)_u \cdot \sum_{q \in Q} u_q \cdot \nu_q \\ &= \textstyle\sum_{u \in Q'} \sum_{q \in Q} h_{\mu'}(\xi)_u \cdot u_q \cdot \nu_q \\ &= \textstyle\sum_{q \in Q} h_\mu(\xi)_q \cdot \nu_q = ([\![M]\!], \xi) \ . \end{aligned}$$

■

Observation 5.4 *Let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $u_1, \ldots, u_k \in A^Q$, and $a_1, \ldots, a_k \in A$. If $a_1, \ldots, a_k \in \{0, 1\}$ or A is commutative, we have that $\mu(\sigma)(a_1 \cdot u_1, \ldots, a_k \cdot u_k) = a_1 \cdot \ldots \cdot a_k \cdot \mu(\sigma)(u_1, \ldots, u_k)$.*

Proof. By elementary calculations. ■

5.2 Minimality

The following theorem, which corresponds to Theorem 3 of [KM05], shows that maximal factorizations generate the smallest bu-det wta among all bu-det wta which are obtained by factorization. Note that this does not mean that there is no smaller equivalent bu-det wta.

Theorem 5.5 *Let A be commutative and let (f, g) and $(\tilde{f}, \tilde{g})$ be factorizations such that (f, g) is maximal. Moreover, let $\det_{(f,g)}(M) = (Q', \mu', \nu')$ and $\det_{(\tilde{f},\tilde{g})}(M) = (\tilde{Q}, \tilde{\mu}, \tilde{\nu})$. Then $Q' = f(\tilde{Q})$; hence $|Q'| \leq |\tilde{Q}|$, and if $\det_{(\tilde{f},\tilde{g})}(M)$ is a wta, then so is $\det_{(f,g)}(M)$.*

Proof. We first show the case that $|Q| = 1$. We can identify A^Q with A. Since (f, g) is maximal, we have that $f(A \setminus \{0\}) = \{f(1)\}$, and since $Q' \neq \emptyset$, $\tilde{Q} \neq \emptyset$, and $\tilde{Q} \subseteq A \setminus \{0\}$, we obtain that $Q' = \{f(1)\} = f(\tilde{Q})$.

Now let $|Q| > 1$. By Lemma 4.4, A is zero-divisor free. Note that $(\star)$ for every $u \in A^Q \setminus \{\tilde{0}\}$, we have $\tilde{g}(u) \cdot \tilde{f}(u) = u = g(u) \cdot f(u)$, and by applying f we obtain $f(\tilde{f}(u)) = f(u) = f(f(u))$ because (f, g) is maximal.

We begin with the proof of $f(\tilde{Q}) \subseteq Q'$. Using Obs. 4.5, it suffices to prove the following statement by induction on i: for every $i \in \mathbb{N}$, $f(\tilde{Q}_i) \subseteq Q'$. To this end, let $i \in \mathbb{N}$ and $\tilde{u} \in f(\tilde{Q}_{i+1})$. Then there are $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\tilde{u}_1, \ldots, \tilde{u}_k \in \tilde{Q}_i$ such that $\tilde{u} = f(\tilde{f}(\mu(\sigma)(\tilde{u}_1, \ldots, \tilde{u}_k)))$. Hence

$$\begin{aligned} \tilde{u} &= f(\tilde{f}(\mu(\sigma)(\tilde{u}_1, \ldots, \tilde{u}_k))) \\ &= f(\mu(\sigma)(\tilde{u}_1, \ldots, \tilde{u}_k)) && (\star) \\ &= f(\mu(\sigma)(f(\tilde{u}_1), \ldots, f(\tilde{u}_k))) && \text{(Lemma 5.6)} \\ &\in Q' \,. && \text{(induction hypothesis, def. of } Q'\text{)} \end{aligned}$$

Now we prove $Q' \subseteq f(\tilde{Q})$. Using Obs. 4.5 again, it suffices to prove the following statement by induction on i: for every $i \in \mathbb{N}$, $Q'_i \subseteq f(\tilde{Q})$. To this end, let $i \in \mathbb{N}$ and $u' \in Q'_{i+1}$. Then there are $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $u'_1, \ldots, u'_k \in Q'_i$ such that $u' = f(\mu(\sigma)(u'_1, \ldots, u'_k))$. By induction hypothesis, there are $\tilde{u}_1, \ldots, \tilde{u}_k \in \tilde{Q}$ such that $u'_i = f(\tilde{u}_i)$ for every $i \in \{1, \ldots, k\}$. Hence

$$\begin{aligned} u' &= f(\mu(\sigma)(f(\tilde{u}_1), \ldots, f(\tilde{u}_k))) \\ &= f(\mu(\sigma)(\tilde{u}_1, \ldots, \tilde{u}_k)) && \text{(Lemma 5.6)} \\ &= f(\tilde{f}(\mu(\sigma)(\tilde{u}_1, \ldots, \tilde{u}_k))) && (\star) \\ &\in f(\tilde{Q}) \,. \end{aligned}$$

■

Lemma 5.6 *Let A be commutative and (f, g) be a maximal factorization. Furthermore, let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $u_1, \ldots, u_k \in A^Q$, and $u'_1, \ldots, u'_k \in A^Q$ such that $u'_i \in \{u_i, f(u_i)\}$ for every $i \in \{1, \ldots, k\}$. Then $\mu(\sigma)(u_1, \ldots, u_k) \neq \tilde{0}$ implies $\mu(\sigma)(u'_1, \ldots, u'_k) \neq \tilde{0}$, and the converse holds if A zero-divisor free. Furthermore, if $\mu(\sigma)(u_1, \ldots, u_k) \neq \tilde{0}$, then $f(\mu(\sigma)(u_1, \ldots, u_k)) = f(\mu(\sigma)(u'_1, \ldots, u'_k))$.*

PROOF. We construct the sequence $a_1, \dots, a_k \in A$ by letting for every $i \in \{1, \dots, k\}$

$$a_i = \begin{cases} g(u_i) & \text{if } u'_i = f(u_i) \ , \\ 1 & \text{otherwise} \ . \end{cases}$$

Clearly then ($\star$)

$$\begin{aligned} \mu(\sigma)(u_1, \dots, u_k) &= \mu(\sigma)(a_1 \cdot u'_1, \dots, a_k \cdot u'_k) && ((f,g) \text{ factorization}) \\ &= a_1 \cdot \ldots \cdot a_k \cdot \mu(\sigma)(u'_1, \dots, u'_k) \ . && (\text{Obs. 5.4}) \end{aligned}$$

First, let $\mu(\sigma)(u_1, \dots, u_k) \neq \tilde{0}$. By ($\star$) also $\mu(\sigma)(u'_1, \dots, u'_k) \neq \tilde{0}$. Applying f to ($\star$) and using that (f, g) is maximal, we obtain that $f(\mu(\sigma)(u_1, \dots, u_k)) = f(\mu(\sigma)(u'_1, \dots, u'_k))$. Second, assume that $\mu(\sigma)(u'_1, \dots, u'_k) \neq \tilde{0}$. Since $a_i \neq 0$ for every $i \in \{1, \dots, k\}$, and since A is zero-divisor free, ($\star$) yields that $\mu(\sigma)(u_1, \dots, u_k) \neq \tilde{0}$. ∎

5.3 Finiteness

In this section, we deal with the sufficient conditions for $det_{(f,g)}(M)$ to be a wta. These conditions are shown in two theorems, the first one dealing with two simple conditions, where (i) provides a formal verification of [MK06] and (ii) transfers the result of [Bor04] to our framework.

Theorem 5.7 *Let (f, g) be the trivial or a maximal factorization, and let A be commutative if (f, g) is maximal. Then $\det_{(f,g)}(M)$ is a wta if (i) M is nonrecursive or (ii) A is locally finite.*

PROOF. Let $det_{(f,g)}(M) = (Q', \mu', \nu')$. By Theorem 5.5, it suffices to consider the case that (f, g) is the trivial factorization. (i) If M is nonrecursive, then $Q' = h_\mu(T_\Sigma)$ is finite. (ii) This case is shown by Lemma 4.7 of [Bor04]. ∎

The second theorem shows the remaining conditions. In particular, (ii) generalizes Theorem 5 of [KM05] from strings to trees.

Theorem 5.8 *Let A be commutative and (f, g) a maximal factorization. Then $\det_{(f,g)}(M)$ is a wta if (i) M is bu-det or (ii) A is extremal and M has the twins property.*

PROOF. Let $det_{(f,g)}(M) = (Q', \mu', \nu')$. By Lemma 5.9, $Q' \subseteq f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\})$. We show that Q' is finite, distinguishing the cases (i) and (ii).

(i) If M is bu-det, then Obs. 3.4 yields that each vector in $h_\mu(T_\Sigma)$ has at most one non-zero component. By this fact and since (f, g) is maximal, we can derive that $|f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\})| \leq |Q|$.

(ii) If A is extremal and M has the twins property, Lemma 5.10 yields that there is a finite set $P \subseteq A^Q$ such that $h_\mu(T_\Sigma) \subseteq \{a \cdot u \mid a \in A, u \in P\}$. We calculate $Q' \subseteq f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\}) \subseteq f(\{a \cdot u \mid a \in A, u \in P\} \setminus \{\tilde{0}\}) \subseteq f(P \setminus \{\tilde{0}\})$ because (f, g) is maximal. Hence, Q' is finite. ∎

As can be seen from our construction, the state set Q' of $det_{(f,g)}(M)$ is obtained by alternatingly applying $\mu(\sigma)$ and f. This alternation can be avoided if (f, g) is a maximal factorization. In that case, one can first apply h_μ, which evaluates operations of the form $\mu(\sigma)$, and then apply f only once. This is expressed in the following lemma, which corresponds to Lemma 2 of [KM05].

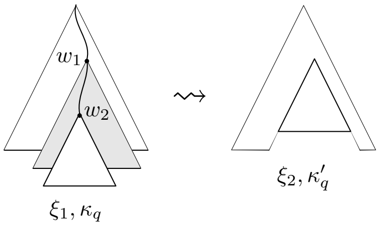


Figure 5: Cutting out the slice starting at w_1 and ending at w_2.

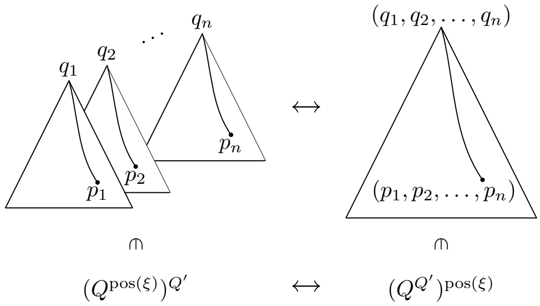


Figure 6: Illustration of $\mathcal{C}'(Q')$.

Lemma 5.9 *Let A be commutative, (f, g) a maximal factorization, and $\det_{(f,g)}(M) = (Q', \mu', \nu')$. Then $Q' \subseteq f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\})$.*

Proof. We first show the case that $|Q| = 1$. Then we can identify A^Q with A, and we obtain that $Q' = \{f(1)\}$. By assumption, there is an $\alpha \in \Sigma^{(0)}$ such that $\mu(\alpha)() \neq 0$. Hence, $h_\mu(\alpha) \neq 0$ and $f(h_\mu(\alpha)) = f(1) \in f(h_\mu(T_\Sigma) \setminus \{0\})$.

Now let $|Q| > 1$. By Lemma 4.4, A is zero-divisor free. Using Obs. 4.5, it suffices to prove the following statement by induction on i: for every $i \in \mathbb{N}$, $Q'_i \subseteq f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\})$. For $i = 0$, this is trivially true. Let $i \in \mathbb{N}$ and $u \in Q'_{i+1}$. Then there are $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $u_1, \ldots, u_k \in Q'_i$ such that $u = f(\mu(\sigma)(u_1, \ldots, u_k))$. By induction hypothesis, there are $\xi_1, \ldots, \xi_k \in T_\Sigma$ such that $u_i = f(h_\mu(\xi_i))$ for every $i \in \{1, \ldots, k\}$. Hence

$$\begin{aligned} u &= f(\mu(\sigma)(u_1, \ldots, u_k)) \\ &= f(\mu(\sigma)(f(h_\mu(\xi_1)), \ldots, f(h_\mu(\xi_k)))) \\ &= f(\mu(\sigma)(h_\mu(\xi_1), \ldots, h_\mu(\xi_k))) && \text{(Lemma 5.6)} \\ &= f(h_\mu(\sigma(\xi_1, \ldots, \xi_k))) \\ &\in f(h_\mu(T_\Sigma) \setminus \{\tilde{0}\}) . \end{aligned}$$

■

The following lemma corresponds to part of the proof of Theorem 5 of [KM05].

Lemma 5.10 *If A is a commutative, extremal semiring and M has the twins property, then there is a finite set $P \subseteq A^Q$ with $h_\mu(T_\Sigma) \subseteq \{a \cdot u \mid a \in A, u \in P\}$.*

We begin by sketching the proof idea. Let $\xi_1 \in T_\Sigma$. Since A is extremal, for every $q \in Q$ there is a $\kappa_q \in R^q(\xi_1)$ such that $h_\mu(\xi_1)_q = wt(\kappa_q)$. If ξ_1 is sufficiently "large", then we find positions w_1 and w_2 such that $w_1 < w_2$ and $\kappa_q(w_1) = \kappa_q(w_2)$ for every $q \in Q$. Provided that we have chosen the family of runs $(\kappa_q \mid q \in Q)$ in a suitable manner, the twins property guarantees that each run in this family assigns the same weight, say a_1, to the "slice" of ξ starting at position w_1 and ending at position w_2 (depicted as the shaded area in Fig. 5). We can remove this slice from ξ_1, obtaining the smaller tree ξ_2 and family of runs $(\kappa'_q \mid q \in Q)$ on ξ_2 with $wt(\kappa_q) = a_1 \cdot wt(\kappa'_q)$. This procedure can be iterated a finite number of times, yielding the trees $\xi_1, \ldots, \xi_n$ and weights $a_1, \ldots, a_{n-1}$, where ξ_n is in a finite set of "small" trees (giving rise to a finite set P of vectors).

Before we can elaborate on the proof idea and prove Lemma 5.10 formally, we need some preliminaries. Let $\xi \in T_\Sigma$, $\xi' \in T_\Sigma \cup C_\Sigma$, and $w \in pos(\xi)$. We denote by $\xi[\xi']_w$ the tree (in

$T_\Sigma \cup C_\Sigma$) obtained from ξ by substituting the subtree rooted at w by ξ'. For every $\kappa \in R(\xi)$ and $\kappa' \in R(\xi')$, we extend the notions $\xi'|_w$ and $\xi[z]_w$ to $\kappa'|_w$ and $\kappa[z]_w$ in the obvious way, i.e., for every $w' \in pos(\xi'|_w)$, we set $\kappa'|_w(w') = \kappa'(ww')$, and for every $w' \in pos(\xi[z]_w)$, we set $\kappa[z]_w(w') = \kappa(w')$. Moreover, for every $q \in Q$, $\xi \in T_\Sigma(\{z\})$, and $\kappa \in R^q(\xi)$ we set $R(\kappa) = R^{q,q}(\xi)$.

The following definition formalizes the notions used for our proof idea, where $\mathcal{C}'$ contains the intermediate results of the process of cutting out the slices, φ assigns a weight vector to each intermediate result, and U assigns to each intermediate result the set of position pairs (w_1, w_2) which are suited for cutting. Special care is needed for runs with weight 0 because the twins property does not apply to them. This is achieved in the following definition by considering subsets Q' of Q. In addition, we define a subset $\mathcal{C} \subset \mathcal{C}'$ which shows an important invariant of the cutting process.

Definition 5.11 *Let $Q' \subseteq Q$. Then we define $\mathcal{C}'(Q') = \{(\xi,\kappa) \mid \xi \in T_\Sigma, \kappa \in R(\xi)^{Q'}, \forall q \in Q' \colon \kappa_q \in R^q(\xi), wt(\kappa_q) \neq 0\}$. We set $\mathcal{C}' = \bigcup_{Q' \subseteq Q} \mathcal{C}'(Q')$. In the following, we define the mapping $\varphi : \mathcal{C}' \to A^Q$, the family $(U(\xi,\kappa) \mid (\xi,\kappa) \in \mathcal{C}'(Q'))$, and the set $\mathcal{C}(Q') \subseteq \mathcal{C}'(Q')$. To this end, let $(\xi,\kappa) \in \mathcal{C}'(Q')$. We will often identify this pair with κ. Then*

- *For every $q \in Q$ we set $\varphi(\kappa)_q = \mathrm{wt}(\kappa_q)$ if $q \in Q'$, otherwise we set $\varphi(\kappa)_q = 0$.*
- *We define $U(\kappa)$ to be the set of all pairs $(w_1, w_2) \in pos(\xi) \times pos(\xi)$ such that $w_1 < w_2$ and for every $q \in Q'$ we have $\kappa_q(w_1) = \kappa_q(w_2)$.*
- *We have $\kappa \in \mathcal{C}(Q')$ iff for every $(w_1, w_2) \in U(\kappa)$ and $q \in Q'$ we have $\mathrm{wt}(\kappa_q|_{w_1}) = \mathrm{wt}(R(\kappa_q|_{w_1}))$. We set $\mathcal{C} = \bigcup_{Q' \subseteq Q} \mathcal{C}(Q')$.* □

We make a remark concerning intuition. Let $(\xi,\kappa) \in \mathcal{C}'$. We feel that for understanding proofs, it is best to view κ as an element of $(Q^{Q'})^{pos(\xi)}$ (see Fig. 6). This is reasonable because $R(\xi)^{Q'} = (Q^{pos(\xi)})^{Q'}$, and there is a natural bijection between $(Q^{pos(\xi)})^{Q'}$ and $(Q^{Q'})^{pos(\xi)}$.

The following lemma shows the first argument of our proof idea, namely that there is a κ such that $(\xi,\kappa) \in \mathcal{C}$ and $\varphi(\kappa) = h_\mu(\xi)$.

Lemma 5.12 *Let A be a commutative, extremal semiring and $\xi \in T_\Sigma$. Then there is a κ such that $(\xi,\kappa) \in \mathcal{C}$ and $\varphi(\kappa) = h_\mu(\xi)$.*

Proof. We begin by showing the following, more general statement by induction on ξ: for every $\xi \in T_\Sigma$ and $q \in Q$ there is a $\kappa \in R^q(\xi)$ such that the statement $P(\xi,\kappa)$ holds, where $P(\xi,\kappa)$ iff for every $w \in pos(\xi)$ we have $\mathrm{wt}(\kappa|_w) = \mathrm{wt}(R(\kappa|_w))$.

Let $\xi = \sigma(\xi_1,\ldots,\xi_k)$. By Observation 5.13 there is a $\kappa' \in R^q(\xi)$ such that $\mathrm{wt}(\kappa') = \mathrm{wt}(R^q(\xi))$. By induction hypothesis, there are $\kappa_1 \in R(\kappa|_1)$, …, $\kappa_k \in R(\kappa|_k)$ such that for every $i \in \{1,\ldots,k\}$ we have $P(\xi_i,\kappa_i)$. Now we construct $\kappa \in R^q(\xi)$ as follows: $\kappa(\varepsilon) = \kappa'(\varepsilon)$ and $\kappa(i \cdot w) = \kappa_i(w)$ for every $i \in \{1,\ldots,k\}$ and $w \in pos(\xi_i)$. It is easy to see that $P(\xi,\kappa)$ holds.

Finally, let $Q' = \{q \in Q \mid h_\mu(\xi)_q \neq 0\}$. We construct $\kappa \in R(\xi)^{Q'}$ as follows: for every $q \in Q'$ let $\kappa_q \in R^q(\xi)$ such that $P(\xi,\kappa)$ holds. By [FV09, Sect. 3.2], we have $\varphi(\kappa) = h_\mu(\xi)$. ■

Observation 5.13 *Let A be an extremal semiring, $\xi \in T_\Sigma$, and $R \subseteq R(\xi)$. Then there is a $\kappa \in R$ such that $\mathrm{wt}(\kappa) = \mathrm{wt}(R)$.*

The next lemma and its proof show how one slice can be cut out.

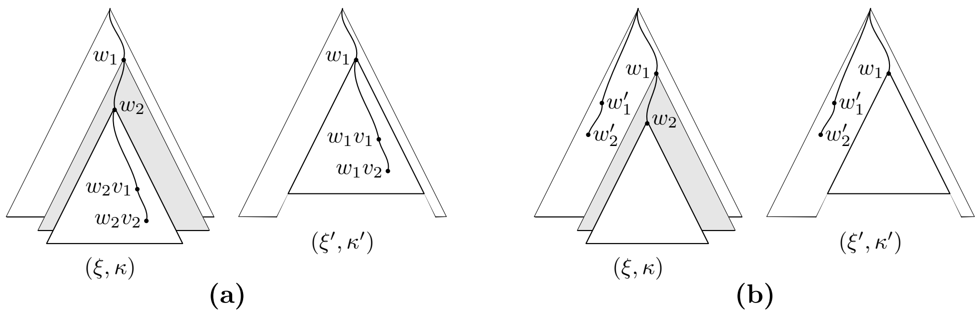


Figure 7: Two cases for $(w'_1, w'_2) \in U(\kappa')$.

Lemma 5.14 *Let A be a commutative, extremal semiring and M have the twins property. Moreover, let $Q' \subseteq Q$ and $(\xi, \kappa) \in \mathcal{C}(Q')$ such that $U(\kappa) \neq \emptyset$. Then there are $(\xi', \kappa') \in \mathcal{C}(Q')$ and $a \in A$ such that $\varphi(\kappa) = a \cdot \varphi(\kappa')$ and $|U(\kappa')| < |U(\kappa)|$.*

Proof. Since $U(\kappa) \neq \emptyset$, there is a pair $(w_1, w_2) \in U(\kappa)$ such that for every $(w'_1, w'_2) \in U(\kappa)$, if $w'_1 \leq w_1$, then $w'_1 = w_1$. We construct $\xi' = \xi[\xi|_{w_2}]_{w_1}$ and for every $q \in Q'$ and $w \in pos(\xi')$, we set $\kappa'_q(w) = \kappa_q(w_2 v)$ if $w = w_1 v$ and $\kappa'_q(w) = \kappa_q(w)$ otherwise. Finally, if $Q' = \emptyset$, we set $a = 0$, otherwise we choose some $q' \in Q'$ and set $a = wt(R(\kappa_{q'}[z]_{w_2}|_{w_1}))$. Now we show that (1) $(\xi', \kappa') \in \mathcal{C}(Q')$, (2) $\varphi(\kappa) = a \cdot \varphi(\kappa')$, and (3) $|U(\kappa')| < |U(\kappa)|$.

We begin with Statement 1. It is easy to see that $(\xi', \kappa') \in \mathcal{C}'(Q')$. Now let $(w'_1, w'_2) \in U(\kappa')$ and $q \in Q'$. We show that $wt(\kappa'_q|_{w'_1}) = wt(R(\kappa'_q|_{w'_1}))$. Note that $w'_1 \not< w_1$. We distinguish two cases.

Case 1 (cf. Fig. 7(a)): There are $v_1, v_2 \in \mathbb{N}^*$ such that $w'_1 = w_1 \cdot v_1$ and $w'_2 = w_1 \cdot v_2$. Since $\kappa'_q|_{w_1} = \kappa_q|_{w_2}$ for every $q \in Q'$, we obtain that $(w_2 v_1, w_2 v_2) \in U(\kappa)$. Hence

$$wt(\kappa'_q|_{w'_1}) = wt(\kappa_q|_{w_2 v_1}) = wt(R(\kappa_q|_{w_2 v_1})) = wt(R(\kappa'_q|_{w'_1})) \ .$$

Case 2 (cf. Fig. 7(b)): Otherwise, $\kappa'_q|_{w'_1} = \kappa_q|_{w_1}$. Thus $(w'_1, w'_2) \in U(\kappa)$ and

$$wt(\kappa'_q|_{w'_1}) = wt(\kappa_q|_{w'_1}) = wt(R(\kappa_q|_{w'_1})) = wt(R(\kappa'_q|_{w'_1})) \ .$$

Now we show Statement 2 for the non-trivial case, i.e., $Q' \neq \emptyset$. Let $q \in Q'$. Then

$$\begin{aligned}
& wt(\kappa_q) \\
&= wt(\kappa_q[z]_{w_1}) \cdot wt(\kappa_q[z]_{w_2}|_{w_1}) \cdot wt(\kappa_q|_{w_2}) && \text{(comm.)} \\
&= wt(\kappa_q[z]_{w_1}) \cdot wt(R(\kappa_q[z]_{w_2}|_{w_1})) \cdot wt(\kappa_q|_{w_2}) && \text{(Observation 5.15)} \\
&= wt(\kappa_q[z]_{w_1}) \cdot wt(R(\kappa_{q'}[z]_{w_2}|_{w_1})) \cdot wt(\kappa_q|_{w_2}) && (\star) \\
&= a \cdot wt(\kappa_q[z]_{w_1}) \cdot wt(\kappa_q|_{w_2}) && \text{(comm.)} \\
&= a \cdot wt(\kappa'_q) \ .
\end{aligned}$$

At $(\star)$, we have used the twins property. We show that this is possible. By definition we have $wt(\kappa_q) \neq 0$ and $wt(\kappa_{q'}) \neq 0$. Hence, also $wt(\kappa_q[z]_{w_2}|_{w_1}) \cdot wt(\kappa_q|_{w_2}) \neq 0$ and $wt(\kappa_{q'}[z]_{w_2}|_{w_1}) \cdot wt(\kappa_{q'}|_{w_2}) \neq 0$, and using the fact that A is extremal, and hence, zero-sum free, we obtain $wt(R(\kappa_q[z]_{w_2}|_{w_1})) \neq 0$, $wt(R(\kappa_{q'}[z]_{w_2}|_{w_1})) \neq 0$, $wt(R(\kappa_q|_{w_2})) \neq 0$, and $wt(R(\kappa_{q'}|_{w_2})) \neq 0$.

Finally, for Statement 3 we remark that κ' is obtained from κ by removing the cycle (w_1, w_2), and that this process does not introduce new cycles. Hence $|U(\kappa')| \leq |U(\kappa)|$. ∎

Observation 5.15 *Let A be a commutative, extremal semiring. Moreover, let $\xi \in T_\Sigma$, $w \in pos(\xi)$, and $\kappa \in R(\xi)$ such that $wt(\kappa) = wt(R(\kappa))$. Then $wt(\kappa[z]_w) \cdot wt(\kappa|_w) = wt(R(\kappa[z]_w)) \cdot wt(\kappa|_w)$.*

Proof. We have

$$\begin{aligned}
& wt(R(\kappa[z]_w)) \cdot wt(\kappa|_w) \\
&= \left(\textstyle\sum_{\nu \in R(\kappa[z]_w)} wt(\nu)\right) \cdot wt(\kappa|_w) \\
&= \textstyle\sum_{\nu \in R(\kappa[z]_w)} wt(\nu) \cdot wt(\kappa|_w) && \text{(distributivity)} \\
&= wt(\kappa) && (\star) \\
&= wt(\kappa[z]_w) \cdot wt(\kappa|_w) \;. && \text{(comm.)}
\end{aligned}$$

Using commutativity, we see that the set of summands on the left-hand side of $(\star)$ is a subset of $\{wt(\nu) \mid \nu \in R(\kappa)\}$ and that it contains $wt(\kappa)$. Since $wt(R(\kappa)) = wt(\kappa)$ and since A is extremal, we can now easily deduce that $(\star)$ holds. ∎

The following lemma is used for the proof that our cutting process can only end in a finite set of trees.

Lemma 5.16 *There is a finite set $\mathcal{F} \subseteq \mathcal{C}'$ such that $U^{-1}(\emptyset) \subseteq \mathcal{F}$.*

Proof. We construct $\mathcal{F} = \{\kappa \in \mathcal{C}' \mid \forall w \in pos(\kappa) \colon |w| < |Q|^{|Q|}\}$. Let $\kappa \in \mathcal{C}' \setminus \mathcal{F}$. We show that $U(\kappa) \neq \emptyset$ follows. First of all, there is a $Q' \subseteq Q$ such that $\kappa \in \mathcal{C}'(Q')$. Since $\kappa \notin \mathcal{F}$, there is a $w \in pos(\kappa)$ such that $|w| \geq |Q|^{|Q'|}$. Hence, there are $k \in \mathbb{N}$, $w_1, \ldots, w_k \in \mathbb{N}^*$, and $u_1, \ldots, u_k \in Q^{Q'}$ such that $k > |Q|^{|Q'|}$, $w_i \in pos(\kappa)$ for every $i \in \{1, \ldots, k\}$, $w_1 < w_2 < \ldots < w_k$, and $\kappa_q|_{w_i} \in R^{(u_i)_q}$ for every $i \in \{1, \ldots, k\}$ and $q \in Q'$. By the pigeon-hole principle, there are $i, j \in \{1, \ldots, k\}$ such that $i < j$ and $u_i = u_j$. Then, however, $(w_i, w_j) \in U(\kappa)$. ∎

Proof (of Lemma 5.10). By Lemma 5.16, there is a set $\mathcal{F} \subseteq \mathcal{C}'$ such that $U^{-1}(\emptyset) \subseteq \mathcal{F}$. We set $P = \varphi(\mathcal{F})$.

Now let $\xi \in T_\Sigma$. Let $n \in \mathbb{N}$ be maximal such that there are $\kappa_1, \ldots, \kappa_n \in \mathcal{C}$ and $a_1, \ldots, a_n \in A$ such that $\varphi(\kappa_1) = h_\mu(\xi)$, $\varphi(\kappa_1) = a_i \cdot \varphi(\kappa_i)$ for every $i \in \{1, \ldots, n\}$, and $|U(\kappa_{i+1})| < |U(\kappa_i)|$ for every $i \in \{1, \ldots, n-1\}$. We claim that (1) $n > 0$ and (2) $\kappa_n \in \mathcal{F}$, which allows us to derive

$$h_\mu(\xi) = \varphi(\kappa_1) = a_n \cdot \varphi(\kappa_n) \in \{a \cdot u \mid a \in A, u \in P\} \;.$$

Statement 1 follows from Lemma 5.12 if we set $a_1 = 1$.

Finally, we prove Statement 2. Assume that $U(\kappa_n) \neq \emptyset$. By Lemma 5.14, there are κ' and a' such that $\varphi(\kappa_n) = a' \cdot \varphi(\kappa')$ and $|U(\kappa')| < |U(\kappa)|$. Using $\kappa_{n+1} = \kappa'$ and $a_{n+1} = a' \cdot a_n$, we see that n was not maximal. Hence, $U(\kappa_n) = \emptyset$, and by Lemma 5.16, $\kappa_n \in \mathcal{F}$. ∎

5.4 Decidability

In this section, we show that the twins property is decidable for a decidable subclass of wta called *cycle unambiguous*. This result is inspired by a similar one for the string case found in [AM03, Theorem 5]. The following definition is also adapted from [AM03, Sect. 2.1]. The wta M is called *cycle unambiguous* if for every $q \in Q$ and $\zeta \in C_\Sigma$ there is at most one $\kappa \in R^{q,q}(\zeta)$ such that $wt(\kappa) \neq 0$. For instance, the wta of Example 3.1 is cycle unambiguous.

We will need the mapping $ht : T_\Sigma \to \mathbb{N}$ associating to every tree $\xi \in T_\Sigma$ its height, where $ht(\xi) = 0$ if $\xi = \alpha$ for some $\alpha \in \Sigma^{(0)}$, and $ht(\xi) = 1 + \max\{ht(\xi_i) \mid i \in \{1, \ldots, k\}\}$ if $\xi = \sigma(\xi_1, \ldots, \xi_k)$.

Theorem 5.17 *The twins property for cycle-unambiguous wta over a commutative zero-sum-free and zero-divisor-free semiring is decidable.*

Proof. Let A be a commutative zero-sum-free and zero-divisor-free semiring and let $M = (Q, \mu, \nu)$ be a cycle-unambiguous wta over Σ and A.

Let the property P be obtained from the twins property by adding the restriction that $ht(\zeta \cdot \xi) < 3 \cdot |Q|^2$. We show that $(\star)$ M has the twins property iff it has property P. Since the property P can be decided, the statement of the theorem follows.

Clearly, the direction "$\Rightarrow$" of $(\star)$ is trivial. We show the other direction by contradiction. Assume that M has property P, but not the twins property. Then there are $p, q \in Q$, $\zeta \in C_\Sigma$, $\xi \in T_\Sigma$ such that

$$ht(\zeta \cdot \xi) \geq 3 \cdot |Q|^2, \tag{1}$$

$$wt(R^{p,p}(\zeta)) \neq 0, wt(R^p(\xi)) \neq 0, \tag{2}$$

$$wt(R^{q,q}(\zeta)) \neq 0, wt(R^q(\xi)) \neq 0, \text{ and} \tag{3}$$

$$wt(R^{p,p}(\zeta)) \neq wt(R^{q,q}(\zeta)) \ . \tag{4}$$

First, we show that we can assume

$$ht(\xi) < |Q|^2 \ . \tag{5}$$

Roughly speaking, we argue that if (5) does not hold, this is only due to our choice of ξ. To this end, assume that we have $\neg$(5). We will construct a tree ξ' such that $|pos(\xi')| < |pos(\xi)|$ and (1′)–(3′) hold, which are obtained from the original statements by priming ξ. By (2) and (3), there are runs $\kappa_p \in R^p(\xi)$ and $\kappa_q \in R^q(\xi)$ such that

$$wt(\kappa_p) \neq 0 \quad \text{and} \quad wt(\kappa_q) \neq 0 \ . \tag{6}$$

By $\neg$(5), there are positions $w_1, w_2 \in pos(\xi)$ such that $w_1 < w_2$, $\kappa_p(w_1) = \kappa_p(w_2)$, and $\kappa_q(w_1) = \kappa_q(w_2)$. We construct the tree ξ' by cutting out the slice between w_1 and w_2, i.e., $\xi' = \xi[\xi|_{w_2}]_{w_1}$, and we obtain the runs $\kappa'_p \in R^p(\xi')$ and $\kappa'_q \in R^q(\xi')$ accordingly. Clearly, $wt(\kappa'_p) \neq 0$ and $wt(\kappa'_q) \neq 0$ because otherwise (6) would be violated. Since A is zero-sum free, also $wt(R^p(\xi')) \neq 0$ and $wt(R^q(\xi')) \neq 0$, which shows (2′) and (3′). Property P (applied to ξ', ζ, p, and q), (2′), and (3′) imply (1′). Iterating this process shows that we can indeed assume (5).

Now we prepare the second step. By (1) and (5), we obtain

$$ht(\zeta) \geq 2 \cdot |Q|^2 \ . \tag{7}$$

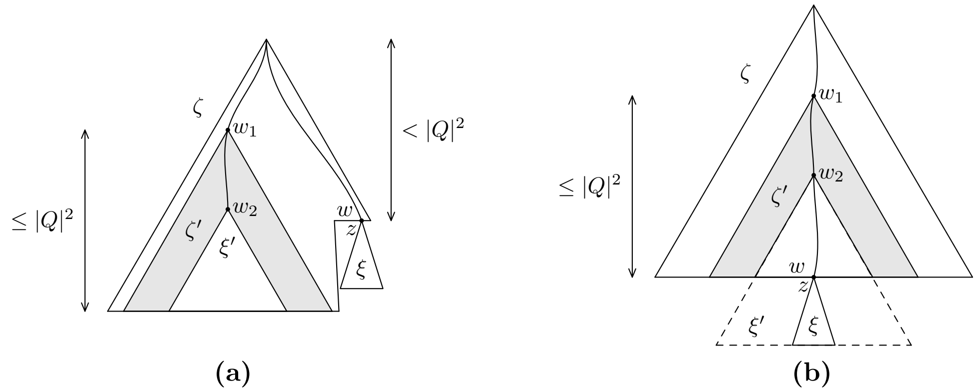


Figure 8: Two cases for $|w|$.

Since M is cycle unambiguous, the (non-zero) weights $wt(R^{p,p}(\zeta))$ and $wt(R^{q,q}(\zeta))$ can not be spread over several runs; thus, there are $\kappa_1 \in R^{p,p}(\zeta)$ and $\kappa_2 \in R^{q,q}(\zeta)$ such that

$$wt(R^{p,p}(\zeta)) = wt(\kappa_1) \quad \text{and} \quad wt(R^{q,q}(\zeta)) = wt(\kappa_2)\ . \tag{8}$$

Let $w \in pos(\zeta)$ such that $\zeta(w) = z$. We define the set $U \subseteq pos(\zeta) \times pos(\zeta)$ as the set of all pairs (w_1, w_2) such that (i) $w_1 < w_2$, (ii) $w_1 \leq w$ implies $w_2 \leq w$, (iii) $\kappa_1(w_1) = \kappa_1(w_2)$ and $\kappa_2(w_1) = \kappa_2(w_2)$, and (iv) the weights of the slices of κ_1 and κ_2 between w_1 and w_2 coincide, i.e., $wt(\kappa_1[z]_{w_2}|_{w_1}) = wt(\kappa_2[z]_{w_2}|_{w_1})$.

Second, we show that we can assume that $U = \emptyset$. Roughly speaking, we argue that if $U \neq \emptyset$, this is only due to our choice of ζ, κ_1, and κ_2. To this end, assume that $U \neq \emptyset$. Hence, there are $w_1, w_2 \in pos(\zeta)$ such that (i)–(iv) hold. We will construct a context ζ' and runs $\kappa'_1 \in R^{p,p}(\zeta')$ and $\kappa'_2 \in R^{q,q}(\zeta')$ such that $|U'| < |U|$ where U' is defined for ζ', κ'_1, and κ'_2 just as U is for ζ, κ_1, and κ_2, and the Statements (1′)–(4′) hold, which are obtained from the original statements by priming ζ. We define Statement (8′) in the same way, however also priming the runs.

Now we construct $\zeta' = \zeta[\zeta|_{w_2}]_{w_1}$, and κ'_1 and κ'_2 are obtained accordingly. We have that $wt(\kappa'_1) \neq 0$ and $wt(\kappa'_2) \neq 0$ because otherwise, $wt(\kappa_1) = 0$ and $wt(\kappa_2) = 0$, and Statements (2), (3), or (8) would be violated. Hence, and since M is cycle unambiguous, we obtain (8′), which shows (2′) and (3′). Statement (4′) follows from (4), (8), the construction of κ'_1 and κ'_2, (iv), and (8′). Then, property P (applied to ξ, ζ', p, and q) and (2′)–(4′) imply (1′). Note that $\zeta' \in C_\Sigma$ by (ii), and that $|U'| < |U|$ because the cycle (w_1, w_2) is cut out and no further cycle is inserted. Hence, iteration of this process yields that we can indeed assume $U = \emptyset$.

Third, we give states p', q', a context ζ', and a tree ξ' such that $\neg$(1′) and (2′)–(4′) hold, where the primed versions are obtained from the original statements by priming q, p, ζ, and ξ. Note that this then contradicts the statement that M has property P. We distinguish two cases.

Case 1 (cf. Fig. 8(a)): if $|w| < |Q|^2$, then by (7) there are positions $(w_1, w_2) \in pos(\zeta)$ such that $w_1 < w_2$, $w_1 \nleq w$, $ht(\zeta|_{w_1}) \leq |Q|^2$, $\kappa_1(w_1) = \kappa_1(w_2)$, and $\kappa_2(w_1) = \kappa_2(w_2)$. We set $\zeta' = \zeta[z]_{w_2}|_{w_1}$, $\xi' = \zeta|_{w_2}$, $p' = \kappa_1(w_1)$, and $q' = \kappa_2(w_1)$. By construction, we have $\neg$(1′). Since $U = \emptyset$, we have that $wt(\kappa_1[z]_{w_2}|_{w_1}) \neq wt(\kappa_2[z]_{w_2}|_{w_1})$, and both quantities are not zero because otherwise $wt(\kappa_1) = 0$ and $wt(\kappa_2) = 0$. Hence, and since M is cycle-unambiguous,

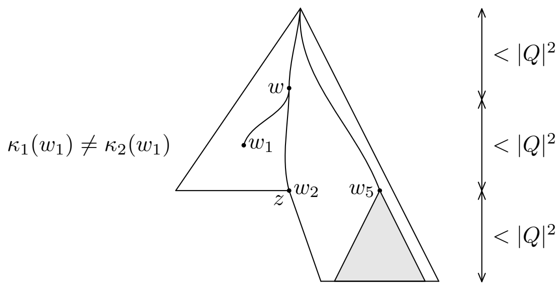


Figure 9: Proof illustration.

we conclude (4′). Since $wt(\kappa_1|_{w_2}) \neq 0$ and $wt(\kappa_2|_{w_2}) \neq 0$, and since A is zero-sum free, we obtain that $wt(R^{p'}(\xi')) \neq 0$ and $wt(R^{q'}(\xi')) \neq 0$. Statements (2′) and (3′) follow.

Case 2 (cf. Fig. 8(b)): if $|w| \geq |Q|^2$, then there are positions $(w_1, w_2) \in pos(\zeta)$ such that $w_1 < w_2 \leq w$, $ht(\zeta|_{w_1}) \leq |Q|^2$, $\kappa_1(w_1) = \kappa_1(w_2)$, and $\kappa_2(w_1) = \kappa_2(w_2)$. We set $\zeta' = \zeta[z]_{w_2}|_{w_1}$, $\xi' = (\zeta \cdot \xi)|_{w_2}$, $p' = \kappa_1(w_1)$, and $q' = \kappa_2(w_1)$. The rest of the proof is verbatim as for Case 1, with the remark that for the proof of $wt(R^{q'}(\xi')) \neq 0$, we need that A is zero-divisor free. ∎

The following lemma shows that the property "cycle unambiguous" is decidable.

Lemma 5.18 *The wta M is cycle unambiguous iff for every $q \in Q$ and $\zeta \in C_\Sigma$ the following condition holds: if $ht(\zeta) < 3 \cdot |Q|^2$, then there is at most one $\kappa \in R^{q,q}(\zeta)$ such that $wt(\kappa) \neq 0$.*

Proof. The "⇒" direction is trivial. We show the "⇐" direction by contradiction. Figure 9 illustrates this proof. Assume that the right-hand side holds (abbreviated by (⋆) in the sequel) but that M is not cycle unambiguous. Hence, there are $q \in Q$, $\zeta \in C_\Sigma$, and $\kappa_1, \kappa_2 \in R^{q,q}(\zeta)$ such that

$$ht(\zeta) \geq 3 \cdot |Q|^2, \tag{1}$$

$$\kappa_1 \neq \kappa_2, \tag{2}$$

$$wt(\kappa_1) \neq 0 \quad \text{and} \quad wt(\kappa_2) \neq 0 \,. \tag{3}$$

By (2), there is a position $w_1 \in pos(\zeta)$ such that $\kappa_1(w_1) \neq \kappa_2(w_1)$. Moreover, let $w_2 \in pos(\zeta)$ such that $\zeta(w_2) = z$, and let w be the longest common prefix of w_1 and w_2.

First, we argue that we can assume that $|w| < |Q|^2$, $|w_1| < 2 \cdot |Q|^2$, and $|w_2| < 2 \cdot |Q|^2$. To this end, first assume that $|w| \geq |Q|^2$. By the pigeonhole principle, there are positions w_3 and w_4 such that $w_3 < w_4 < w$, $\kappa_1(w_3) = \kappa_1(w_4)$, and $\kappa_2(w_3) = \kappa_2(w_4)$, that is, $w = w_4 w'$ for some w'. We obtain a new context ζ' and new runs κ_1' and κ_2' by cutting from the old ones the slice between w_3 and w_4. Note that $\zeta' \in C_\Sigma$ and $\kappa_1' \neq \kappa_2'$ because the part below w (now $w_3 w'$) has been retained, and that $wt(\kappa_1') \neq 0$ and $wt(\kappa_2') \neq 0$ because otherwise we find a contradiction to (3). Since $|w_3 w'| < |w_4 w'|$, repeated application of this procedure shows that we can indeed assume $|w| < |Q|^2$. The argumentation for the assumptions on w_1 and w_2 is done in exactly the same manner—there we find cutting points in between w and w_1 (respectively, w_2).

Second, we show that we can assume ¬(1), which is a contradiction to (⋆). By (1), there is a position $w_5 \in pos(\zeta)$ such that $|w_5| = 2 \cdot |Q|^2$ and $ht(\zeta|_{w_5}) \geq |Q|^2$. Note that by our first step, we can assume that $w_5 \not\leq w_1$ and $w_5 \not\leq w_2$. Consequently, $\zeta|_{w_5} \in T_\Sigma$. By the pigeonhole principle, we find positions $w_6, w_7 \in pos(\zeta|_{w_5})$ such that $w_6 < w_7$, $\kappa_1(w_5w_6) = \kappa_1(w_5w_7)$, and $\kappa_2(w_5w_6) = \kappa_2(w_5w_7)$. We obtain ζ', κ_1', and κ_2' by cutting out the slice between w_5w_6 and w_5w_7. Again, we find that $\zeta' \in C_\Sigma$, $\kappa_1' \neq \kappa_2'$, $wt(\kappa_1') \neq 0$, and $wt(\kappa_2') \neq 0$. Since $|pos(\kappa'|_{w_5})| < |pos(\kappa|_{w_5})|$, repeated application of this procedure does show that we can indeed assume $\neg(1)$. ■

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